

BUILDING HIGHLY CONDITIONAL ALMOST GREEDY AND QUASI-GREEDY BASES IN BANACH SPACES

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ABSTRACT. It is known that for a conditional quasi-greedy basis \mathcal{B} in a Banach space \mathbb{X} , the associated sequence $(k_m[\mathcal{B}])_{m=1}^\infty$ of its conditionality constants verifies the estimate $k_m[\mathcal{B}] = \mathcal{O}(\log m)$ and that if the reverse inequality $\log m = \mathcal{O}(k_m[\mathcal{B}])$ holds then \mathbb{X} is non-superreflexive. Indeed, it is known that a quasi-greedy basis in a superreflexive quasi-Banach space fulfils the estimate $k_m[\mathcal{B}] = \mathcal{O}(\log m)^{1-\epsilon}$ for some $\epsilon > 0$. However, in the existing literature one finds very few instances of spaces possessing quasi-greedy basis with conditionality constants “as large as possible.” Our goal in this article is to fill this gap. To that end we enhance and exploit a technique developed by Dilworth et al. in [16] and craft a wealth of new examples of both non-superreflexive classical Banach spaces having quasi-greedy bases \mathcal{B} with $k_m[\mathcal{B}] = \mathcal{O}(\log m)$ and superreflexive classical Banach spaces having for every $\epsilon > 0$ quasi-greedy bases \mathcal{B} with $k_m[\mathcal{B}] = \mathcal{O}(\log m)^{1-\epsilon}$. Moreover, in most cases those bases will be almost greedy.

1. INTRODUCTION

Let $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$ be a (Schauder) basis for a Banach space \mathbb{X} and let $\mathcal{B}^* = (\mathbf{x}_j^*)_{j=1}^\infty$ be its sequence of coordinate functionals. By default all our bases (and basic sequences) will be assumed to be *semi-normalized*, i.e., $0 < \inf_{j \in \mathbb{N}} \|\mathbf{x}_j\| \leq \sup_{j \in \mathbb{N}} \|\mathbf{x}_j\| < \infty$. We will denote by $S_m[\mathcal{B}, \mathbb{X}]$

2010 *Mathematics Subject Classification.* 46B15, 41A65.

Key words and phrases. Thresholding Greedy algorithm, conditional basis, conditionality constants, quasi-greedy basis, almost greedy basis, subsymmetric basis, superreflexivity, sequence spaces, Banach spaces.

F. Albiac and J.L. Ansorena are partially supported by the Spanish Research Grant *Análisis Vectorial, Multilineal y Aplicaciones*, reference number MTM2014-53009-P. F. Albiac also acknowledges the support of Spanish Research Grant *Operators, lattices, and structure of Banach spaces*, with reference MTM2016-76808-P. S. J. Dilworth was supported by the National Science Foundation under Grant Number DMS-1361461. S. J. Dilworth and Denka Kutzarova were supported by the Workshop in Analysis and Probability at Texas A&M University in 2017.

the m -th partial sum projection with respect to \mathcal{B} , i.e.,

$$S_m[\mathcal{B}, \mathbb{X}](f) = \sum_{j=1}^m \mathbf{x}_j^*(f) \mathbf{x}_j, \quad f \in \mathbb{X}.$$

Given a subset A of \mathbb{N} , the *coordinate projection* on A is (when well defined) the linear operator $S_A[\mathcal{B}, \mathbb{X}]: \mathbb{X} \rightarrow \mathbb{X}$ given by

$$f \mapsto \sum_{j \in A} \mathbf{x}_j^*(f) \mathbf{x}_j.$$

A basis $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$ is *bimototone* if $\|S_A[\mathcal{B}, \mathbb{X}]\| \leq 1$ for every interval A of integers in \mathbb{N} . Every semi-normalized basis in a Banach space \mathbb{X} is normalized (i.e., $\|\mathbf{x}_j\| = 1$ for all j) and bimonotone under a suitable renorming of \mathbb{X} .

A basis \mathcal{B} is *unconditional* if and only if $\sup_{A \text{ finite}} \|S_A[\mathcal{B}, \mathbb{X}]\| < \infty$. Thus, in some sense, the *conditionality* of \mathcal{B} can be measured in terms of the growth of the sequence

$$k_m[\mathcal{B}, \mathbb{X}] := \sup_{|A| \leq m} \|S_A[\mathcal{B}, \mathbb{X}]\|, \quad m \in \mathbb{N}.$$

Recall that a basis $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$ is said to be *quasi-greedy* if it is semi-normalized and there is a constant C such that

$$\|f - S_F[\mathcal{B}, \mathbb{X}](f)\| \leq C\|f\| \quad (1.1)$$

whenever $f \in \mathbb{X}$ and $F \subseteq \mathbb{N}$ are such that $|\mathbf{x}_j^*(f)| \leq |\mathbf{x}_k^*(f)|$ for all $j \in \mathbb{N} \setminus F$ and all $k \in F$. The least constant C such that (1.1) holds is known as the quasi-greedy constant of the basis (see [4, Remark 4.2]).

The next theorem summarizes the connection that exists between superreflexivity and the conditionality constants of quasi-greedy bases. Recall that a space \mathbb{X} is said to be *superreflexive* if every Banach space finitely representable in \mathbb{X} is reflexive.

Theorem 1.1 (see [6, 7, 16, 24]). *Let \mathbb{X} be a Banach space.*

- (a) *If \mathcal{B} is a quasi-greedy basis for \mathbb{X} then $k_m[\mathcal{B}, \mathbb{X}] \lesssim \log m$ for $m \geq 2$.*
- (b) *If \mathbb{X} is non-superreflexive then there is a quasi-greedy basis \mathcal{B} for a Banach space \mathbb{Y} finitely representable in \mathbb{X} with $k_m[\mathcal{B}, \mathbb{Y}] \gtrsim \log m$ for $m \geq 2$.*
- (c) *If \mathbb{X} is superreflexive and \mathcal{B} is quasi-greedy then there is $0 < a < 1$ such that $k_m[\mathcal{B}] \lesssim (\log m)^a$ for $m \geq 2$.*
- (d) *For every $0 < a < 1$ there is a quasi-greedy basis \mathcal{B} for a Banach space \mathbb{Y} (namely, $\mathbb{Y} = \ell_2$) finitely representable in \mathbb{X} with $k_m[\mathcal{B}, \mathbb{Y}] \gtrsim (\log m)^a$ for $m \geq 2$.*

Theorem 1.1 characterizes both the superreflexivity and the lack of superreflexivity of a Banach space \mathbb{X} in terms of the growth of the conditionality constants of quasi-greedy bases. It could be argued that the quasi-greedy bases whose existence is guaranteed in parts (b) and (d) lie outside the space \mathbb{X} , and that, although this approach is consistent when dealing with “super” properties, in truth it does not tackle the question of the existence of a quasi-greedy basis with large conditionality constants in the space \mathbb{X} itself! Hence this discussion naturally leads to the following two questions relative to a given Banach space:

Question A. Pick a non-superreflexive Banach space \mathbb{X} with a quasi-greedy basis. Is there a quasi-greedy basis \mathcal{B} for \mathbb{X} with $k_m[\mathcal{B}, \mathbb{X}] \approx \log m$ for $m \geq 2$?

Question B. Pick a non-superreflexive Banach space \mathbb{X} with a quasi-greedy basis. Given $0 < a < 1$, is there a quasi-greedy basis \mathcal{B} for \mathbb{X} with $k_m[\mathcal{B}, \mathbb{X}] \gtrsim (\log m)^a$ for $m \geq 2$?

Questions A and B can be regarded as a development of the query initiated by Konyagin and Telmyakov in 1999 [30] of finding conditional quasi-greedy bases in general Banach spaces, and which has evolved towards the more specific quest of finding quasi-greedy bases “as conditional as possible.” The reader will find a detailed account of this process in the papers [7, 16, 19, 23, 28, 40].

Let us outline the state of the art of those two questions. Garrigós and Wojtaszczyk proved in [24] that Question B has a positive answer for $\mathbb{X} = \ell_p$, $1 < p < \infty$. As for Question A, it is known that Lindenstrauss’ basic sequence in ℓ_1 , the Haar system in $BV(\mathbb{R}^d)$ for $d \geq 2$, and the unit-vector system in the Konyagin-Telmyakov space $KT(\infty, p)$ for $1 < p < \infty$, are all quasi-greedy basic sequences with conditionality constants as large as possible (see [12, 23]). Moreover, in [24] it is proved that the answer to Question A is positive for $\ell_1 \oplus \ell_2 \oplus c_0$, and in [6] that the same holds true for mixed-norm spaces of the form $(\bigoplus_{n=1}^{\infty} \ell_1^n)_q$ ($1 < q < \infty$), providing this way the first-known examples of reflexive Banach spaces having quasi-greedy bases with conditionality constants as large as possible. More recently, the authors constructed in [7] the first-known examples of Banach spaces of nontrivial type and nontrivial cotype for which the answer to Question A is positive. These spaces are $\mathcal{W}_{p,q}^0 \oplus \mathcal{W}_{p,q}^0 \oplus \ell_2$, $1 < p, q < \infty$, where $\mathcal{W}_{p,q}^0$ and $\mathcal{W}_{p,q}$ are the interpolation spaces

$$\mathcal{W}_{p,q}^0 = (v_1^0, c_0)_{\theta,q}, \quad \text{and} \quad \mathcal{W}_{p,q} = (v_1, \ell_\infty)_{\theta,q}, \quad p = 1/(1 - \theta), \quad (1.2)$$

defined from the space of sequences of bounded variation

$$v_1 = \left\{ (a_j)_{j=1}^\infty : |a_1| + \sum_{j=2}^\infty |a_j - a_{j-1}| < \infty \right\},$$

and the subspace v_1^0 of v_1 resulting from the intersection of v_1 with c_0 . Here and throughout this paper, $(\mathbb{X}_0, \mathbb{X}_1)_{\theta, q}$ denotes the Banach space obtained by applying the real interpolation method with indices θ and q to the Banach couple $(\mathbb{X}_0, \mathbb{X}_1)$. Let us recall that, in light of [16, Theorem 8.5], the only quasi-greedy basis for a Banach space whose dual has the Grothendieck Theorem Property is the unit vector basis of c_0 . Consequently, the answer to Question A is negative for c_0 . Any other separable \mathcal{L}_∞ -space, despite having a basis (see [27, Theorem 5.1]), does not have a quasi-greedy basis.

In this article we develop the necessary machinery that permits to extend the scant list of known Banach spaces for which the answer either to Question A or to Question B is positive. Moreover, for a wide class of Banach spaces the examples of bases we provide are not only quasi-greedy but are *almost greedy*. Recall that a basis $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$ for a Banach space \mathbb{X} is *almost greedy* if there is a constant $C \geq 1$ such that for any $f \in \mathbb{X}$ we have

$$\|f - S_F[\mathcal{B}, \mathbb{X}](f)\| \leq C \|f - S_A[\mathcal{B}, \mathbb{X}](f)\|$$

whenever $|A| \leq |F| < \infty$, and $|\mathbf{x}_j^*(f)| \leq |\mathbf{x}_k^*(f)|$ for any $j \in \mathbb{N} \setminus F$ and $k \in F$. Almost greedy bases were characterized in [17] as those bases that are simultaneously quasi-greedy and democratic. In fact, in this characterization democracy can be replaced with super-democracy. A basis \mathcal{B} is *super-democratic* if there is a sequence $(\lambda_m)_{m=1}^\infty$ such that for any $A \subseteq \mathbb{N}$ finite and any $(\varepsilon_j)_{j \in A}$ sequence of signs,

$$\left\| \sum_{j \in A} \varepsilon_j \mathbf{x}_j \right\| \approx \lambda_{|A|}.$$

In this case $(\lambda_m)_{m=1}^\infty$ is equivalent to the fundamental function of \mathcal{B} , defined by

$$\varphi_m[\mathcal{B}, \mathbb{X}] = \sup_{|A| \leq m} \left\| \sum_{j \in A} \mathbf{x}_j \right\|, \quad m \in \mathbb{N}.$$

A more demanding concept than super-democracy is that of bi-democracy. A basis \mathcal{B} is said to be *bi-democratic* if

$$\varphi_m[\mathcal{B}, \mathbb{X}] \varphi_m[\mathcal{B}^*, \mathbb{Y}] \lesssim m \quad \text{for all } m \in \mathbb{N},$$

where \mathbb{Y} is the closed linear span in \mathbb{X}^* of the basic sequence \mathcal{B}^* .

Our study includes, among other spaces, the finite direct sums

$$D_{p,q} := \begin{cases} \ell_p \oplus \ell_q & \text{if } 1 \leq p, q < \infty, \\ \ell_p \oplus c_0 & \text{if } 1 \leq p < \infty \text{ and } q = 0, \end{cases}$$

the matrix spaces

$$Z_{p,q} := \begin{cases} \ell_q(\ell_p) & \text{if } 1 \leq p, q < \infty, \\ c_0(\ell_p) & \text{if } 1 \leq p < \infty \text{ and } q = 0, \\ \ell_q(c_0) & \text{if } p = 0 \text{ and } 1 \leq q < \infty, \end{cases}$$

and the mixed-norm spaces of the family

$$B_{p,q} := (\oplus_{n=1}^{\infty} \ell_p^n)_q, \quad p \in [1, \infty], \quad q \in \{0\} \cup [1, \infty), \quad q \neq 0 \text{ when } p = \infty.$$

We use $(\oplus_{n=1}^{\infty} \mathbb{X}_n)_q$ to denote the direct sum of the Banach spaces \mathbb{X}_n in the ℓ_q -sense (in the c_0 -sense if $q = 0$).

In this article we shall prove that the answer to Question A is positive for all non-superreflexive spaces in the aforementioned list. We also show that in all superreflexive spaces in the above list the answer to Question B is positive. These results will appear in Section 4. Previously, in Sections 2 and 3, we introduce the tools that we will use to achieve our goal.

Throughout this article we follow standard Banach space terminology and notation as can be found, e.g., in [8]. Next we single out the notation and terminology that is more commonly employed. We deal with real or complex Banach spaces, and \mathbb{F} will denote the underlying scalar field. A weight will be a sequence of positive scalars. Given families of positive real numbers $(\alpha_i)_{i \in I}$ and $(\beta_i)_{i \in I}$, the symbol $\alpha_i \lesssim \beta_i$ for $i \in I$ means that $\sup_{i \in I} \alpha_i / \beta_i < \infty$. If $\alpha_i \lesssim \beta_i$ and $\beta_i \lesssim \alpha_i$ for $i \in I$ we say that $(\alpha_i)_{i \in I}$ and $(\beta_i)_{i \in I}$ are equivalent, and we write $\alpha_i \approx \beta_i$ for $i \in I$. Applied to Banach spaces, the symbol $\mathbb{X} \approx \mathbb{Y}$ means that the spaces \mathbb{X} and \mathbb{Y} are isomorphic, while the symbol $\mathbb{X} \lesssim_c \mathbb{Y}$ means that \mathbb{X} is isomorphic to a complemented subspace of \mathbb{Y} . The norm of a linear operator $T: \mathbb{X} \rightarrow \mathbb{Y}$ will be denoted either by $\|T\|_{\mathbb{X} \rightarrow \mathbb{Y}}$ or, when the Banach spaces \mathbb{X} and \mathbb{Y} are clear from context, simply by $\|T\|$. Given families of Banach spaces $(\mathbb{X}_i)_{i \in I}$ and $(\mathbb{Y}_i)_{i \in I}$, the symbol $\mathbb{X}_i \lesssim_c \mathbb{Y}_i$ for $i \in I$ means that the spaces \mathbb{X}_i are uniformly isomorphic to complemented subspaces of \mathbb{Y}_i , i.e., there are linear operators $L_i: \mathbb{X}_i \rightarrow \mathbb{Y}_i$, $T_i: \mathbb{Y}_i \rightarrow \mathbb{X}_i$ such that $T_i \circ L_i = \text{Id}_{\mathbb{X}_i}$ and $\sup_i \|T_i\| \|L_i\| < \infty$. Similarly the symbol $\mathbb{X}_i \approx \mathbb{Y}_i$ for $i \in I$ means that the Banach-Mazur distance from \mathbb{X}_i to \mathbb{Y}_i is uniformly bounded. We write $\mathbb{X} \oplus \mathbb{Y}$ for the Cartesian product of the Banach spaces \mathbb{X} and \mathbb{Y} endowed with the norm

$$\|(x, y)\|_{\mathbb{X} \oplus \mathbb{Y}} = \|x\| + \|y\|, \quad x \in \mathbb{X}, \quad y \in \mathbb{Y}.$$

As is customary, we put $\delta_{j,k} = 1$ if $j = k$ and $\delta_{j,k} = 0$ otherwise. For $j \in \mathbb{N}$, the j -th unit vector refers to the vector $\mathbf{e}_j = (\delta_{j,k})_{k=1}^\infty$, and the sequence $\mathcal{E} := (\mathbf{e}_j)_{j=1}^\infty$ will be called the *unit-vector system*. We will denote by \mathbf{e}_j^* the j -th coordinate functional, defined on $\mathbb{F}^\mathbb{N}$ by $(a_k)_{k=1}^\infty \mapsto a_j$, and by $S_A: \mathbb{F}^\mathbb{N} \rightarrow \mathbb{F}^\mathbb{N}$ the coordinate projection on a set $A \subseteq \mathbb{N}$. For $m \in \mathbb{N}$, S_m will be the coordinate projection on the set $\{1, \dots, m\}$.

The linear span of a family $(x_i)_{i \in I}$ in a Banach space will be denoted by $\langle x_i: i \in I \rangle$, and its closed linear span by $[x_i: i \in I]$. Any other more specific notation will be introduced on the spot as needed.

2. PRELIMINARY RESULTS

Most of the ideas behind the results we include in this preliminary section have appeared more or less explicitly in the literature before. Nonetheless, for the sake of clarity and completeness, we shall include the statements of the results we need in the form that best suits our purposes as well as the sketches of their proofs.

Definition 2.1. Given a basis \mathcal{B} for a Banach space \mathbb{X} , we define the sequence $(L_m[\mathcal{B}, \mathbb{X}])_{m=1}^\infty$ by

$$L_m[\mathcal{B}, \mathbb{X}] = \sup \left\{ \frac{\|S_A[\mathcal{B}, \mathbb{X}](f)\|}{\|f\|} : \max(\text{supp}(f)) \leq m, A \subseteq \mathbb{N} \right\}.$$

When the space \mathbb{X} is clear from context we will simply write $L_m[\mathcal{B}]$. Likewise, we will drop \mathbb{X} from the notation of all the other concepts that we introduced in Section 1 involving a basis \mathcal{B} for a Banach space.

Notice that \mathcal{B} is unconditional if and only if $\sup_m L_m[\mathcal{B}] < \infty$. Hence, the growth of the sequence $(L_m[\mathcal{B}])_{m=1}^\infty$ provides also a measure of the conditionality of the basis. Since $L_m[\mathcal{B}] \leq k_m[\mathcal{B}]$ for all $m \in \mathbb{N}$, any result establishing that the size of the members of the sequence $(L_m[\mathcal{B}])_{m=1}^\infty$ is large, is a stronger statement than the corresponding one enunciated in terms of $(k_m[\mathcal{B}])_{m=1}^\infty$.

The papers [7, 24] draw attention to the fact that, in some cases, the sequence $(L_m[\mathcal{B}])_{m=1}^\infty$ is more fit than the “usual” sequence of conditionality constants $(k_m[\mathcal{B}])_{m=1}^\infty$ for transferring conditionality properties from a given basis to a basis constructed from it. This is the reason why we establish all the instrumental results of this section in terms of the conditionality constants $(L_m[\mathcal{B}])_{m=1}^\infty$. Notice that, in contrast to $(k_m[\mathcal{B}])_{m=1}^\infty$, the sequence $(L_m[\mathcal{B}])_{m=1}^\infty$ is not necessarily doubling. This leads us to use a doubling function in our statements. Recall that a function $\delta: [0, \infty) \rightarrow [0, \infty)$ is said to be *doubling* if for some non-negative constant C one has $\delta(2t) \leq C\delta(t)$ for all $t \geq 0$.

Given sequences $\mathcal{B}_0 = (\mathbf{x}_j)_{j=1}^\infty$ and $\mathcal{B}_1 = (\mathbf{y}_j)_{j=1}^\infty$ in Banach spaces \mathbb{X} and \mathbb{Y} , respectively, their *direct sum* $\mathcal{B}_0 \oplus \mathcal{B}_1$ will be the sequence in $\mathbb{X} \times \mathbb{Y}$ given by

$$\mathcal{B}_0 \oplus \mathcal{B}_1 = ((\mathbf{x}_1, 0), (0, \mathbf{y}_1), (\mathbf{x}_2, 0), (0, \mathbf{y}_2), \dots, ((\mathbf{x}_j, 0), (0, \mathbf{y}_j), \dots).$$

Lemma 2.2 (cf. [23]). *Suppose that $\mathcal{B}_0 = (\mathbf{x}_j)_{j=1}^\infty$ and $\mathcal{B}_1 = (\mathbf{y}_j)_{j=1}^\infty$ are bases for some Banach spaces \mathbb{X} and \mathbb{Y} , respectively. Assume that for some non-decreasing doubling function $\delta: [0, \infty) \rightarrow [0, \infty)$ we have $L_m[\mathcal{B}_0] \gtrsim \delta(m)$ for $m \in \mathbb{N}$. Then $\mathcal{B}_0 \oplus \mathcal{B}_1$ is a basis for $\mathbb{X} \oplus \mathbb{Y}$ with $L_m[\mathcal{B}_0 \oplus \mathcal{B}_1] \gtrsim \delta(m)$ for $m \in \mathbb{N}$. Moreover:*

- (a) *If \mathcal{B}_0 and \mathcal{B}_1 are quasi-greedy so is $\mathcal{B}_0 \oplus \mathcal{B}_1$.*
- (b) *If \mathcal{B}_0 and \mathcal{B}_1 are super-democratic, both with fundamental function equivalent to $(\lambda_m)_{m=1}^\infty$, then $\mathcal{B}_0 \oplus \mathcal{B}_1$ is super-democratic with fundamental function equivalent to $(\lambda_m)_{m=1}^\infty$.*

Proof. It is similar to the proof of [23, Proposition 6.1], so we omit it. \square

Our next lemma follows an idea from [40] for constructing quasi-greedy bases. To state it properly, it will be convenient to introduce some additional notation. Given $N \in \mathbb{N}$ and a Banach space $\mathbb{X} \subseteq \mathbb{F}^\mathbb{N}$ for which the unit-vector system is a basis, we will denote by $\mathbb{X}^{(N)}$ the N -dimensional space $[\mathbf{e}_j: 1 \leq j \leq N]$ regarded as a subspace of \mathbb{X} . As is customary, we will write

$$\ell_p^N := \ell_p^{(N)}, \quad 1 \leq p < \infty, \quad \text{and} \quad \ell_\infty^N := c_0^{(N)}.$$

More generally, given a basis $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$ for a Banach space \mathbb{X} and $N \in \mathbb{N}$ we will consider the closed linear span of the truncated finite sequence $(\mathbf{x}_j)_{j=1}^N$, i.e.,

$$\mathbb{X}^{(N)}[\mathcal{B}] = [\mathbf{x}_j: 1 \leq j \leq N].$$

Let $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$ be a basis in a Banach space \mathbb{X} . Given a sequence of positive integers $(N_n)_{n=1}^\infty$ we define a sequence $(\mathbf{z}_k)_{k=1}^\infty$ in the space $\mathbb{X}^\mathbb{N}$, which we will denote by $\bigoplus_{n=1}^\infty (\mathbf{x}_j)_{j=1}^{N_n}$, by

$$\mathbf{z}_k = (\underbrace{0, \dots, 0}_{r-1 \text{ times}}, \mathbf{x}_j, 0, \dots, 0, \dots),$$

where, for a given $k \in \mathbb{N}$, the integers r and j are univocally determined by the relations $k = j + \sum_{n=1}^{r-1} N_n$ and $1 \leq j \leq N_r$.

Lemma 2.3. *Suppose that $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$ is a basis for a Banach space \mathbb{X} with $L_m[\mathcal{B}] \gtrsim \delta(m)$ for $m \in \mathbb{N}$, for some non-decreasing doubling*

function $\delta: [0, \infty) \rightarrow [0, \infty)$. Let $(N_n)_{n=1}^\infty$ be a sequence of positive integers verifying

$$M_r := \sum_{n=1}^r N_n \lesssim N_{r+1} \text{ for } r \in \mathbb{N}.$$

Then

- (a) The sequence $\mathcal{B}_0 = \bigoplus_{n=1}^\infty (\mathbf{x}_j)_{j=1}^{N_n}$ is a basis for the Banach space $(\bigoplus_{n=1}^\infty \mathbb{X}^{(N_n)}[\mathcal{B}])_p$, $p \in \{0\} \cup [1, \infty)$, with $L_m[\mathcal{B}_0] \gtrsim \delta(m)$ for $m \in \mathbb{N}$.
- (b) If \mathcal{B} is quasi-greedy so is \mathcal{B}_0 .
- (c) If \mathcal{B} is super-democratic with fundamental function equivalent to $(m^{1/p})_{m=1}^\infty$ for some $1 \leq p < \infty$, so is \mathcal{B}_0 .

Proof. It is clear that \mathcal{B}_0 is a quasi-greedy basis with the same quasi-greedy constant as \mathcal{B} , hence we need only take care of obtaining an estimate for $(L_m[\mathcal{B}_0])_{m=1}^\infty$. Let C_1 be such that $C_1 L_m[\mathcal{B}] \geq \delta(m)$ for all $m \in \mathbb{N}$. Let $C_2 > 1$ be such that $M_r \leq C_2 N_r$ for all $r \in \mathbb{N}$. Since δ is doubling there is a constant C_3 such that $C_3 \delta(m) \geq \delta(2C_2 m)$ for all $m \in \mathbb{N}$. Given $m \geq M_1$, pick $r \in \mathbb{N}$ such that $M_r \leq m < M_{r+1}$. We have

$$\begin{aligned} C_1 C_3 L_m[\mathcal{B}_0] &\geq C_1 C_3 \max\{L_{N_r}[\mathcal{B}], L_{m-M_r}[\mathcal{B}]\} \\ &\geq C_3 \max\{\delta(N_r), \delta(m - M_r)\} \\ &\geq \max\{\delta(2C_2 N_r), \delta(2C_2(m - M_r))\} \\ &\geq \max\{\delta(2M_r), \delta(2m - 2M_r)\} \\ &= \delta(\max\{2M_r, 2m - 2M_r\}) \\ &\geq \delta(m), \end{aligned}$$

as desired. \square

The spaces $\mathcal{W}_{p,q}^0$ and $\mathcal{W}_{p,q}$ ($1 < p < \infty$, $1 \leq q < \infty$) defined in (1.2) were introduced and studied by Pisier and Xu [36]. It is verified that $\mathcal{W}_{p,q}^0 \approx \mathcal{W}_{p,q}$. Moreover, when $q > 1$ these spaces have nontrivial type and nontrivial cotype and they are pseudo-reflexive.

Our next proposition is a new addition to the study of Pisier-Xu spaces, which will be used below. Recall that given $1 \leq q < \infty$ and a scalar sequence $\mathbf{w} = (w_n)_{n=1}^\infty$, the Lorentz sequence space $d_q(\mathbf{w})$ consists of all sequences f in c_0 whose non-increasing rearrangement $(a_n^*)_{n=1}^\infty$ verifies

$$\|f\|_{d_q(\mathbf{w})} = \left(\sum_{n=1}^\infty (a_n^*)^q w_n \right)^{1/q} < \infty.$$

In the case when $\mathbf{w} = (n^{q/p-1})_{n=1}^\infty$ for some $1 \leq p < \infty$ we have that $\ell_{p,q} := d_q(\mathbf{w})$ is the classical sequence Lorentz space of indices p and q .

Proposition 2.4. *Let $1 < p < \infty$ and $1 \leq q < \infty$. Then*

$$\ell_{p,q} \lesssim_c \mathcal{W}_{p,q}^0.$$

In fact, $(\mathbf{e}_{2j-1})_{j=1}^\infty$ is a complemented basic sequence isometrically equivalent to the unit vector system in $\ell_{p,q}$.

Proof. Put $p = 1/(1 - \theta)$. Consider the linear maps $L, T: \mathbb{F}^\mathbb{N} \rightarrow \mathbb{F}^\mathbb{N}$ defined by

$$L((a_j)_{j=1}^\infty) = (a_1, 0, a_2, 0, \dots, a_j, 0, a_{j+1}, 0, \dots), \quad (2.1)$$

$$T((a_j)_{j=1}^\infty) = (a_1 - a_2, a_3 - a_4, \dots, a_{2j-1} - a_{2j}, \dots). \quad (2.2)$$

We have $\|L: \ell_1 \rightarrow v_1^0\| \leq 1$, $\|L: c_0 \rightarrow c_0\| \leq 1$, $\|T: v_1^0 \rightarrow \ell_1\| \leq 1$, and $\|T: c_0 \rightarrow c_0\| \leq 2$. Taking into account that

$$(\ell_1, c_0)_{\theta,q} = (\ell_1, \ell_\infty)_{\theta,q} = \ell_{p,q}$$

(see, e.g., [13, Theorem 1.9]), interpolation gives $\|L: \ell_{p,q} \rightarrow \mathcal{W}_{p,q}^0\| \leq 1$ and $\|T: \mathcal{W}_{p,q}^0 \rightarrow \ell_{p,q}\| \leq 2^\theta$. Since $T(L(f)) = f$ for every $f \in \mathbb{F}^\mathbb{N}$ we are done. \square

Corollary 2.5. *Let $1 < p < \infty$ and $1 \leq q < \infty$. Then $\ell_q \lesssim_c \mathcal{W}_{p,q}^0$.*

Proof. In light of Proposition 2.4, it suffices to see that $\ell_q \lesssim_c \ell_{p,q}$. By [33, Proposition 4] we have $\ell_q \lesssim_c \ell_{p,q}$ if $q \leq p$ and $\ell_{q'} \lesssim_c \ell_{p',q'}$ otherwise. We conclude the proof by dualizing (see [10, Theorem 1]). \square

3. THE DILWORTH-KALTON-KUTZAROVA METHOD, REVISITED

Recall that a basis is said to be *subsymmetric* if it is unconditional and equivalent to all of its subsequences. If $(\mathbf{x}_j)_{j=1}^\infty$ is a subsymmetric basis in a Banach space $(\mathbb{S}, \|\cdot\|_\mathbb{S})$ then there is a constant C such that

$$C^{-1} \left\| \sum_{j=1}^\infty a_j \mathbf{x}_j \right\|_\mathbb{S} \leq \left\| \sum_{j=1}^\infty \varepsilon_j a_j \mathbf{x}_{\phi(j)} \right\|_\mathbb{S} \leq C \left\| \sum_{j=1}^\infty a_j \mathbf{x}_j \right\|_\mathbb{S}$$

for all $\sum_{j=1}^\infty a_j \mathbf{x}_j \in \mathbb{S}$, all sequence of signs $(\varepsilon_j)_{j=1}^\infty$, and all increasing maps ϕ , in which case the basis is said to be C -subsymmetric. Every subsymmetric basis is quasi-greedy and super-democratic, hence almost greedy.

A *subsymmetric sequence space* will be a Banach space $\mathbb{S} \subseteq \mathbb{F}^\mathbb{N}$ for which the unit-vector system is a 1-subsymmetric basis. Note that every Banach space equipped with a subsymmetric basis is isomorphic to a subsymmetric sequence space (see, e.g., [11]).

Given $f \in \mathbb{F}^{\mathbb{N}}$ and $A \subseteq \mathbb{N}$ finite, we put

$$\text{Av}(f, A) = \frac{1}{|A|} \left(\sum_{j \in A} a_j \right).$$

The *averaging projection* with respect to a sequence $\sigma = (\sigma_n)_{n=1}^{\infty}$ of disjoint finite subsets of \mathbb{N} is the map $P_{\sigma}: \mathbb{F}^{\mathbb{N}} \rightarrow \mathbb{F}^{\mathbb{N}}$ defined by

$$(a_j)_{j=1}^{\infty} \mapsto (b_k)_{k=1}^{\infty}, \quad b_k = \text{Av}(f, \sigma_n) \text{ if } k \in \sigma_n.$$

We denote by $Q_{\sigma} = \text{Id}_{\mathbb{F}^{\mathbb{N}}} - P_{\sigma}$ its complementary projection. A sequence $(\sigma_n)_{n=1}^{\infty}$ of subsets of \mathbb{N} is said to be *ordered* when

$$\max(\sigma_n) < \min(\sigma_{n+1}), \quad n \in \mathbb{N}.$$

Note that any ordered sequence consists of disjoint finite subsets. Let us recall the following result (see [32, Propostion 3.a.4]).

Theorem 3.1. *Let $(\mathbb{S}, \|\cdot\|_{\mathbb{S}})$ be a subsymmetric sequence space and σ an ordered sequence of subsets of \mathbb{N} . Then P_{σ} is bounded from \mathbb{S} into \mathbb{S} with $\|P_{\sigma}\|_{\mathbb{S} \rightarrow \mathbb{S}} \leq 2$.*

An *ordered partition* of \mathbb{N} will be an ordered sequence $(\sigma_n)_{n=1}^{\infty}$ of subsets of \mathbb{N} with $\mathbb{N} = \cup_n \sigma_n$. Notice that ordered partitions consist of integer intervals. In fact, if $\sigma = (\sigma_n)_{n=1}^{\infty}$ and we let

$$M_r = \sum_{n=1}^r |\sigma_n|, \quad r \in \mathbb{N} \cup \{0\},$$

we have $\sigma_n = [1 + M_{n-1}, M_n]$. Theorem 3.1 immediately gives

$$\|P_{\sigma}(f)\|_{\mathbb{S}} \leq 2\|f\|_{\mathbb{S}}, \quad \|Q_{\sigma}(f)\|_{\mathbb{S}} \leq 3\|f\|_{\mathbb{S}}, \quad f \in c_{00} \quad (3.1)$$

for any ordered partition σ and any subsymmetric sequence space \mathbb{S} . Another consequence of Theorem 3.1 is that subsymmetric bases are bi-democratic. To be precise for any subsymmetric sequence space $(\mathbb{S}, \|\cdot\|_{\mathbb{S}})$ we have (see [32, Proposition 3.a.6])

$$m \leq \left\| \sum_{j=1}^m \mathbf{e}_j \right\|_{\mathbb{S}} \left\| \sum_{j=1}^m \mathbf{e}_j^* \right\|_{\mathbb{S}^*} \leq 2m, \quad m \in \mathbb{N}. \quad (3.2)$$

Remark 3.2. Notice that if \mathcal{B} is a subsymmetric basis for a Banach space \mathbb{S} then $\mathbb{S} \oplus \mathbb{S} \approx \mathbb{S}$. To see this it suffices to consider the subsequences \mathcal{B}_o and \mathcal{B}_e consisting, respectively, of the odd and the even terms of \mathcal{B} . Then, on the one hand, we have that $\mathcal{B}_o \oplus \mathcal{B}_e$ is equivalent to \mathcal{B} and, on the other hand, it is equivalent to $\mathcal{B} \oplus \mathcal{B}$.

Given a subsymmetric sequence space $(\mathbb{S}, \|\cdot\|_{\mathbb{S}})$ and an ordered partition $\sigma = (\sigma_n)_{n=1}^{\infty}$ of \mathbb{N} we will put

$$\Lambda_m = \left\| \sum_{j=1}^m \mathbf{e}_j \right\|_{\mathbb{S}}, \quad \Lambda_m^* = \frac{m}{\Lambda_m}, \quad m \in \mathbb{N}, \quad (3.3)$$

and

$$\mathbf{v}_n = \frac{1}{\Lambda_{|\sigma_n|}} \sum_{j \in \sigma_n} \mathbf{e}_j, \quad \mathbf{v}_n^* = \frac{1}{\Lambda_{|\sigma_n|}^*} \sum_{j \in \sigma_n} \mathbf{e}_j^*, \quad n \in \mathbb{N}. \quad (3.4)$$

We have that $(\mathbf{v}_n)_{n=1}^{\infty}$ is a normalized basic sequence in \mathbb{S} and that

$$P_{\sigma}(f) = \sum_{n=1}^{\infty} \mathbf{v}_n^*(f) \mathbf{v}_n, \quad f \in \mathbb{F}^{\mathbb{N}}.$$

Consequently, by inequality (3.1),

$$\|f\|_{\mathbb{S}} \leq \|Q_{\sigma}(f)\|_{\mathbb{S}} + \left\| \sum_{n=1}^{\infty} \mathbf{v}_n^*(f) \mathbf{v}_n \right\|_{\mathbb{S}} \leq 5\|f\|_{\mathbb{S}}, \quad f \in \mathbb{S}. \quad (3.5)$$

That is, the middle term in (3.5) defines an equivalent norm for \mathbb{S} . Replacing the basic sequence $(\mathbf{v}_n)_{n=1}^{\infty}$ with an arbitrary basis provides a method, invented in [16], for constructing Banach spaces with special types of bases. Let us give a precise description of this method. Suppose $\mathcal{B} = (\mathbf{x}_n)_{n=1}^{\infty}$ is a semi-normalized basis for a Banach space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$. Consider the linear mapping $H: c_{00} \rightarrow c_{00} \times \mathbb{X}$ given by

$$f \mapsto \left(Q_{\sigma}(f), \sum_{n=1}^{\infty} \mathbf{v}_n^*(f) \mathbf{x}_n \right), \quad (3.6)$$

and define a gauge on c_{00} , by

$$\|f\|_{\mathcal{B}, \mathbb{S}, \sigma} = \|H(f)\|_{\mathbb{S} \oplus \mathbb{X}} = \|Q_{\sigma}(f)\|_{\mathbb{S}} + \left\| \sum_{n=1}^{\infty} \mathbf{v}_n^*(f) \mathbf{x}_n \right\|_{\mathbb{X}}.$$

The authors of [16] considered the sequence Banach space obtained by the completion of $(c_{00}, \|\cdot\|_{\mathcal{B}, \mathbb{S}, \sigma})$. In order to justify the validity of this procedure, called for short the *DKK-method* from now on, we need a couple of lemmas.

Lemma 3.3. *Let \mathcal{B} be a (semi-normalized) basis of a Banach space \mathbb{X} . Let \mathbb{S} be a subsymmetric sequence space and σ be an ordered partition of \mathbb{N} . Then there are constants C_1 and C_2 such that*

$$C_1^{-1} \|f\|_{\mathbb{S}} \leq \|f\|_{\mathcal{B}, \mathbb{S}, \sigma} \leq C_2 \|f\|_{\mathbb{S}},$$

for all sequences f supported on σ_k for some $k \in \mathbb{N}$. Moreover, if \mathcal{B} is normalized, then $C_1 = 1$ and $C_2 = 5$.

Proof. Without loss of generality assume that $\mathcal{B} = (\mathbf{x}_n)_{n=1}^\infty$ is normalized. Let \mathbf{v}_n and \mathbf{v}_n^* be for $n \in \mathbb{N}$ as in (3.4). Let $k \in \mathbb{N}$ and $f \in c_{00}$ be such that $\text{supp } f \subseteq \sigma_k$. Since $\mathbf{v}_n^*(f) = 0$ for $n \neq k$ we have

$$\|f\|_{\mathcal{B}, \mathbb{S}, \sigma} = \|Q_\sigma(f)\|_{\mathbb{S}} + |\mathbf{v}_k^*(f)| \|\mathbf{x}_k\|_{\mathbb{X}} = \|Q_\sigma(f)\|_{\mathbb{S}} + \left\| \sum_{n=1}^\infty \mathbf{v}_n^*(f) \mathbf{v}_n \right\|_{\mathbb{S}}.$$

We conclude the proof by appealing to (3.5). \square

Lemma 3.4. *Let \mathbb{S} be a subsymmetric sequence space, and $\sigma = (\sigma_n)_{n=1}^\infty$ be an ordered partition of \mathbb{N} . Let $A \subseteq \mathbb{N}$ finite. With (\mathbf{v}_n^*) as in (3.4) and $B = \cup_{n \in A} \sigma_n$ we have*

$$\mathbf{v}_n^*(S_B(f)) = \begin{cases} \mathbf{v}_n^*(f) & \text{if } n \in A \\ 0 & \text{otherwise,} \end{cases}$$

and

$$S_B(Q_\sigma(f)) = Q_\sigma(S_B(f))$$

for all $f \in c_{00}$.

Proof. The first equality is clear from the definition and implies that $S_B(P_\sigma(f)) = P_\sigma(S_B(f))$. Thus the second identity holds. \square

From Lemma 3.3 and Lemma 3.4 we deduce the existence of a constant C such that

$$|a_k| \leq C \|f\|_{\mathcal{B}, \mathbb{S}, \sigma}$$

for all $k \in \mathbb{N}$ and all $f = (a_j)_{j=1}^\infty \in c_{00}$. We infer that $(c_{00}, \|\cdot\|_{\mathcal{B}, \mathbb{S}, \sigma})$ is a normed space whose completion can be carried out inside c_0 . Thus we can safely give the following definition.

Definition 3.5. Let \mathcal{B} be a basis of a Banach space \mathbb{X} . Let \mathbb{S} be a subsymmetric sequence space and σ be an ordered partition of \mathbb{N} .

- We define the space $\mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma]$ as the completion of $(c_{00}, \|\cdot\|_{\mathcal{B}, \mathbb{S}, \sigma})$ carried out inside c_0 .
- $\mathbb{Y}^{(N)}[\mathcal{B}, \mathbb{S}, \sigma]$ will be the N -dimensional space $[\mathbf{e}_j : 1 \leq j \leq N]$ regarded as a subspace of $\mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma]$.

The next theorem summarizes some early properties of the DKK-method. For the sake of expositional ease we include a sketch of their proof.

Theorem 3.6 (cf. [16]). *Let $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$ be a basis for a Banach space \mathbb{X} , let \mathbb{S} be a subsymmetric sequence space, and $\sigma = (\sigma_n)_{n=1}^\infty$ be an ordered partition of \mathbb{N} . We have:*

- (a) *The unit-vector system $\mathcal{E} = (\mathbf{e}_n)_{n=1}^\infty$ is a semi-normalized basis for $\mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma]$.*

- (b) $\mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma] \approx Q_\sigma(\mathbb{S}) \oplus \mathbb{X}$. In fact, the mapping H in (3.6) extends to an isomorphism from $\mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma]$ onto $Q_\sigma(\mathbb{S}) \oplus \mathbb{X}$.
- (c) If $M_r = \sum_{n=1}^r |\sigma_n|$ for $k \in \mathbb{N}$ then

$$\mathbb{Y}^{(M_r)}[\mathcal{B}, \mathbb{S}, \sigma] \approx Q_\sigma(\mathbb{S}^{(M_r)}) \oplus \mathbb{X}^{(r)}[\mathcal{B}].$$

- (d) If \mathcal{B}' is a basis for a Banach space \mathbb{X}' then (up to an equivalent norm) $\mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma] = \mathbb{Y}[\mathcal{B}', \mathbb{S}, \sigma]$ if and only if the bases \mathcal{B} and \mathcal{B}' are equivalent.

Proof. Set $Q = Q_\sigma$ and let H be as in (3.6). By Lemma 3.4 we have

$$H(S_{M_r}(f)) = (S_{M_r}(Q(f)), S_r[\mathcal{B}](f)), \quad r \in \mathbb{N}, f \in c_{00}.$$

Hence, the partial-sum projections $(S_{M_r})_{r=1}^\infty$ are uniformly bounded. Combining with Lemma 3.3 gives (a).

The mapping H is continuous by definition and so it extends to a continuous map \tilde{H} from $\mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma]$ into $Q(\mathbb{S}) \oplus \mathbb{X}$. Let us prove that \tilde{H} is an onto isomorphism. Put $V = c_{00} \cap Q(\mathbb{S})$, $X = \langle \mathbf{x}_k : k \in \mathbb{N} \rangle$, and let L be the linear map from X into c_{00} defined by $L(\mathbf{x}_n) = \mathbf{v}_n$ for all $n \in \mathbb{N}$. It is straightforward to check that the map

$$G : V \times X \rightarrow c_{00}, \quad (g, h) \mapsto g + L(h)$$

is continuous and verifies $G(H(f)) = f$ for all $f \in c_{00}$, and also $H(G(g, h)) = (g, h)$ for all $g \in V$ and $h \in X$. Hence, in order to obtain (b) it only remains to see that V is a dense subspace of $Q(\mathbb{S})$. Let $f \in Q(\mathbb{S})$ and $\epsilon > 0$. Pick $g \in U$ such $\|f - g\|_{\mathbb{S}} \leq \epsilon/\|Q\|$. Since

$$\|f - Q(g)\|_{\mathbb{S}} = \|Q(f) - Q(g)\|_{\mathbb{S}} \leq \|Q\| \|f - g\|_{\mathbb{S}} \leq \epsilon$$

and $Q(g) \in V$ we are done.

We obtain (c) by restricting the isomorphism \tilde{H} to $\mathbb{Y}^{(M_r)}[\mathcal{B}, \mathbb{S}, \sigma]$.

Assume that $\mathcal{B}' = (\mathbf{x}'_n)_{n=1}^\infty$ is a basis for \mathbb{X}' . Let L' and G' be the operators corresponding, respectively, to L and G when replacing \mathcal{B} with \mathcal{B}' . We have that $\mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma] = \mathbb{Y}[\mathcal{B}', \mathbb{S}, \sigma]$ if and only if $H \circ G'$ extends to an isomorphism from $Q(\mathbb{S}) \oplus \mathbb{X}$ onto $Q(\mathbb{S}) \oplus \mathbb{X}'$. It is straightforward to check that

$$H \left(G' \left(g, \sum_{n=1}^\infty a_n \mathbf{x}_n \right) \right) = \left(g, \sum_{n=1}^\infty a_n \mathbf{x}'_n \right), \quad g \in V, (a_n)_{n=1}^\infty \in c_{00}.$$

Hence, $\mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma] = \mathbb{Y}[\mathcal{B}', \mathbb{S}, \sigma]$ if and only if the mapping

$$\sum_{n=1}^\infty a_n \mathbf{x}_n \mapsto \sum_{n=1}^\infty a_n \mathbf{x}'_n, \quad (a_n)_{n=1}^\infty \in c_{00}$$

extends to an isomorphism from \mathbb{X} onto \mathbb{X}' . □

Part (d) of Theorem 3.6 alerts us that in the case when the basis \mathcal{B} is conditional so is the unit-vector system of $\mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma]$. Indeed, it is possible to obtain a relation between the conditionality constants of both bases.

Lemma 3.7. *Let \mathcal{B} be a basis for a Banach space \mathbb{X} , let \mathbb{S} be a subsymmetric sequence space, and $\sigma = (\sigma_n)_{n=1}^\infty$ be an ordered partition of \mathbb{N} . If $M_r = \sum_{n=1}^r |\sigma_n|$ for all $r \in \mathbb{N}$, then*

$$L_{M_r}[\mathcal{E}, \mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma]] \geq L_r[\mathcal{B}, \mathbb{X}].$$

Proof. Let Λ_n , \mathbf{v}_n , and \mathbf{v}_n^* be defined for all $n \in \mathbb{N}$ as in (3.3) and (3.4). Put $m_0 = 0$. Given a non-null r -tuple $(a_n)_{n=1}^r$ we define a sequence $f = (b_j)_{j=1}^\infty$ by

$$b_j = \begin{cases} a_n / \Lambda_{|\sigma_n|} & \text{if } j \in \sigma_n \text{ for some } 1 \leq n \leq r, \\ 0 & \text{if } j > M_r. \end{cases}$$

We have $\mathbf{v}_n^*(f) = a_n$ for all $n \in \mathbb{N}$, $P_\sigma(f) = f$ and, then, $Q_\sigma(f) = 0$. Consequently, for any $A \subseteq \mathbb{N}$, putting $B = \cup_{n \in A} \sigma_n$, we have

$$L_{M_r}[\mathcal{E}, \mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma]] \geq \frac{\|S_B(f)\|_{\mathcal{B}, \mathbb{S}, \sigma}}{\|f\|_{\mathcal{B}, \mathbb{S}, \sigma}} = \frac{\|S_A(\sum_{n=1}^r a_n \mathbf{x}_n)\|_{\mathbb{X}}}{\|\sum_{n=1}^r a_n \mathbf{x}_n\|_{\mathbb{X}}}.$$

We finish the proof by taking the supremum on $(a_n)_{n=1}^r$. \square

Proposition 3.8. *Let \mathcal{B} be a basis for a Banach space \mathbb{X} , let \mathbb{S} be a subsymmetric sequence space and $\sigma = (\sigma_n)_{n=1}^\infty$ be an ordered partition of \mathbb{N} . Assume that $L_m[\mathcal{B}] \gtrsim \delta(m)$ for $m \in \mathbb{N}$ for some non-decreasing doubling function $\delta: [0, \infty) \rightarrow [0, \infty)$ and that*

$$\log \left(\sum_{n=1}^r |\sigma_n| \right) \lesssim r \text{ for } r \in \mathbb{N}. \quad (3.7)$$

Then $L_m[\mathcal{E}, \mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma]] \gtrsim \delta(\log m)$ for $m \in \mathbb{N}$.

Proof. If $M_r = \sum_{n=1}^r |\sigma_n|$, there is $C_1 > 0$ such that $\log M_r \leq C_1(r-1)$ for all $r \geq 2$. Let $C_2 \in (0, \infty)$ be such that $\delta(C_1 x) \leq C_2 \delta(x)$ for all $x \geq 0$. Let $C_3 \in (0, \infty)$ be such that $\delta(m) \leq C_3 L_m[\mathcal{B}]$ for all $m \in \mathbb{N}$. Given $m \in \mathbb{N}$ with $m \geq M_1$, pick $r \geq 1$ such that $M_r \leq m < M_{r+1}$. Invoking Lemma 3.7 we get

$$\begin{aligned} C_2 C_3 L_m[\mathcal{E}, \mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma]] &\geq C_2 C_3 L_{M_r}[\mathcal{E}, \mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma]] \\ &\geq C_2 C_3 L_r[\mathcal{B}, \mathbb{X}] \\ &\geq C_2 \delta(r) \\ &\geq \delta(C_1 r) \end{aligned}$$

$$\begin{aligned} &\geq \delta(\log M_{r+1}) \\ &\geq \delta(\log m). \end{aligned}$$

□

3.1. Democracy-like properties from the DKK-method. The super-democracy of the unit-vector system in sequence spaces obtained by the DKK-method will be inferred by means of embeddings involving Lorentz sequence spaces.

Given a weight $\mathbf{w} = (w_n)_{n=1}^\infty$, the weak Lorentz sequence space $d_1^\infty(\mathbf{w})$ consists of all sequences $f = (a_n)_{n=1}^\infty \in c_0$ whose non-increasing rearrangement $(a_n^*)_{n=1}^\infty$ verifies

$$\|f\|_{d_1^\infty(\mathbf{w})} = \sup_m a_m^* \sum_{n=1}^m w_n = \sup_{a>0} a \sum_{\substack{j: |a_j|>a}} w_n < \infty.$$

It is well-known that an embedding of the form

$$d_1(\mathbf{w}) \subseteq \mathbb{Y} \subseteq d_1^\infty(\mathbf{w})$$

implies that the unit-vector basis of the sequence space \mathbb{Y} is super-democratic with fundamental function equivalent to $(\sum_{n=1}^m w_n)_{m=1}^\infty$. Conversely, any almost greedy basis fulfils such embeddings (see [1, Theorem 3.1]).

In the case when $(\mathbb{S}, \|\cdot\|_{\mathbb{S}})$ is a subsymmetric sequence space, the sequence $(\Lambda_n)_{n=1}^\infty$ defined as in (3.4) is the fundamental function of the unit-vector system and if $\mathbf{w} = (\Lambda_n - \Lambda_{n-1})_{n=1}^\infty$ then

$$\|f\|_{d_1^\infty(\mathbf{w})} \leq \|f\|_{\mathbb{S}} \leq \|f\|_{d_1(\mathbf{w})} \quad (3.8)$$

(cf. [5, Theorem 6.1]). Note that while $(\Lambda_n)_{n=1}^\infty$ is non-decreasing, the sequence $(\Lambda_n/n)_{n=1}^\infty$ is non-increasing (see [17, comments below Theorem 3.1]). Of course, this monotonicity can be expressed as

$$\frac{\Lambda_m}{m} \leq \frac{\Lambda_n}{n}, \quad m \geq n, \quad (3.9)$$

but also in the form

$$\Lambda_n - \Lambda_{n-1} \leq \frac{\Lambda_n}{n}, \quad n \in \mathbb{N}. \quad (3.10)$$

Theorem 3.9 (cf. [16, Proposition 6.1]). *Let \mathcal{B} be a basis for a Banach space \mathbb{X} , let $(\mathbb{S}, \|\cdot\|_{\mathbb{S}})$ be a subsymmetric sequence space, and $\sigma = (\sigma_n)_{n=1}^\infty$ be an ordered partition of \mathbb{N} . Assume that*

$$M_r := \sum_{n=1}^r |\sigma_n| \lesssim |\sigma_{r+1}| \text{ for } r \in \mathbb{N}. \quad (3.11)$$

Let $\mathbf{w} = (\Lambda_n - \Lambda_{n-1})_{n=1}^\infty$ be as in (3.4) and $\mathbf{w}' = (\Lambda_n/n)_{n=1}^\infty$. Then

$$d_1(\mathbf{w}') \subseteq \mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma] \subseteq d_1^\infty(\mathbf{w})$$

(with continuous embeddings).

Proof. Let $C_\sigma = \sup_r M_r/|\sigma_r|$. Assume that $\mathcal{B} = (\mathbf{x}_n)_{n=1}^\infty$ is a bi-monotone and normalized basis for $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$.

Note that if $j \in \sigma_n$ then, by (3.9),

$$\frac{\Lambda_j}{j} \geq \frac{\Lambda_{M_n}}{M_n} \geq \frac{\Lambda_{|\sigma_n|}}{C_\sigma |\sigma_n|}.$$

If $f = (a_j)_{j=1}^\infty \in c_{00}$,

$$\begin{aligned} \left\| \sum_{n=1}^\infty \mathbf{v}_n^*(f) \mathbf{x}_n \right\|_{\mathbb{X}} &\leq \sum_{n=1}^\infty |\mathbf{v}_n^*(f)| \\ &\leq \sum_{n=1}^\infty \frac{\Lambda_{|\sigma_n|}}{|\sigma_n|} \sum_{j \in \sigma_n} |a_j| \\ &\leq C_\sigma \sum_{j=1}^\infty \frac{\Lambda_j}{j} |a_j|. \end{aligned}$$

Hence, appealing to the rearrangement inequality and to (3.9),

$$\left\| \sum_{n=1}^\infty \mathbf{v}_n^*(f) \mathbf{x}_n \right\|_{\mathbb{X}} \leq C_\sigma \|f\|_{d_1(\mathbf{w}')}.$$

By (3.1), (3.8) and (3.10), we also have

$$\|Q_\sigma(f)\|_{\mathbb{S}} \leq 3\|f\|_{\mathbb{S}} \leq 3\|f\|_{d_1(\mathbf{w})} \leq 3\|f\|_{d_1(\mathbf{w}')}.$$

Combining, we get

$$\|f\|_{\mathcal{B}, \mathbb{S}, \sigma} \leq (3 + C_\sigma) \|f\|_{d_1(\mathbf{w}')}.$$

Let us now look at the lower estimate. Suppose $a > 0$ and let $B := \{k \in \mathbb{N} : |a_k| > a\}$. Put $t = C_\sigma/(1 + C_\sigma)$ and define

$$A = \{n \in \mathbb{N} : |\mathbf{v}_n^*(f)| < t a \Lambda_{|\sigma_n|}\} = \{n \in \mathbb{N} : |\text{Av}(f, \sigma_n)| < t a\}.$$

If $k \in B \cap \sigma_n$ and $n \in A$, we have

$$|\mathbf{e}_k^*(Q_\sigma(f))| = |a_k - \text{Av}(f, \sigma_n)| \geq |a_k| - |\text{Av}(f, \sigma_n)| > (1-t)a = \frac{a}{1 + C_\sigma}.$$

Thus, if $B_0 := B \cap (\cup_{n \in A} \sigma_n)$,

$$\|Q_\sigma(f)\|_{\mathbb{S}} \geq \frac{a \Lambda_{|B_0|}}{1 + C_\sigma}.$$

If m is the largest integer in $\mathbb{N} \setminus A$ we have $B_1 := B \setminus (\cup_{n \in A} \sigma_n) \subseteq \cup_{n=1}^m \sigma_n$. Therefore $|B_1| \leq C_\sigma |\sigma_m|$ and then there is an integer N with $\max\{|B_1|, |\sigma_m|\} \leq N \leq C_\sigma |\sigma_m|$. By (3.9),

$$\Lambda_{|B_1|} \leq \Lambda_N \leq \frac{N}{|\sigma_m|} \Lambda_{|\sigma_m|} \leq \frac{C_\sigma}{ta} |\mathbf{v}_m^*(f)| = \frac{1 + C_\sigma}{a} |\mathbf{v}_m^*(f)|.$$

Combining,

$$\|f\|_{\mathcal{B}, \mathbb{S}, \sigma} \geq \|Q_\sigma(f)\|_{\mathbb{S}} + |\mathbf{v}_m^*(f)| \geq \frac{a}{1 + C_\sigma} (\Lambda_{|B_0|} + \Lambda_{|B_1|}) \geq \frac{a}{1 + C_\sigma} \Lambda_{|B|}.$$

Hence

$$\|f\|_{d_1^\infty(\mathbf{w})} \leq (1 + C_\sigma) \|f\|_{\mathcal{B}, \mathbb{S}, \sigma}.$$

□

Following [17], we say that a weight $(\lambda_m)_{m=1}^\infty$ has the *lower regularity property* (LRP for short) if there a positive integer b such

$$2\lambda_m \leq \lambda_{bm}, \quad m \in \mathbb{N}.$$

We will also need the so-called *upper regularity property* (URP for short). We say that $(\lambda_m)_{m=1}^\infty$ has the URP if there is an integer $b \geq 3$ such that

$$\lambda_{bm} \leq \frac{b}{2} \lambda_m, \quad m \in \mathbb{N}.$$

Corollary 3.10. *Assume that the hypotheses of Theorem 3.9 hold and that $(\Lambda_n)_{n=1}^\infty$ has the LRP. Then*

$$d_1(\mathbf{w}) \subseteq \mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma] \subseteq d_1^\infty(\mathbf{w}).$$

Proof. From the LRP and (3.9) (see, e.g., [1, Lemma 2.12]) we infer that $(\Lambda_n)_{n=1}^\infty$ satisfies the Dini condition

$$\sum_{n=1}^m \frac{\Lambda_n}{n} \leq C_d \Lambda_m \text{ for all } m \in \mathbb{N},$$

for some constant C_d . Hence, $d_1(\mathbf{w}) \subseteq d_1(\mathbf{w}')$. Combining with Theorem 3.9 yields

$$d_1(\mathbf{w}) \subseteq \mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma] \subseteq d_1^\infty(\mathbf{w}).$$

Quantitatively, if $C_\sigma = \sup_r M_r / |\sigma_r|$,

$$\frac{1}{1 + C_\sigma} \|f\|_{d_1^\infty(\mathbf{w})} \leq \|f\|_{\mathcal{B}, \mathbb{S}, \sigma} \leq (3 + C_\sigma) C_d \|f\|_{d_1(\mathbf{w})}.$$

for all $f \in c_{00}$.

□

3.2. Quasi-greediness in bases from the DKK-method. The embeddings provided by Lemma 3.10 are considered by some authors as a property which ensures in a certain sense the optimality of the compression algorithms with respect to the basis (see [21]). The authors of [1] point out that almost greediness is a stronger condition. In this Section we show that an extra property on the symmetric sequence space, namely URP, fills the gap between those two properties.

Lemma 3.11. *Let $(\mathbb{S}, \|\cdot\|_{\mathbb{S}})$ be a subsymmetric sequence space and $\sigma = (\sigma_n)_{n=1}^{\infty}$ be an ordered partition of \mathbb{N} . Let Λ_n^* and \mathbf{v}_n^* for $n \in \mathbb{N}$ be as in (3.3) and (3.4). Then*

$$|\mathbf{v}_n^*(S_A(f))| \leq 2 \frac{\Lambda_{|A|}^*}{\Lambda_{|\sigma_n|}^*} \|S_{\sigma_n}(f)\|_{\mathbb{S}}$$

whenever $n \in \mathbb{N}$, $A \subseteq \sigma_n$ and $f \in c_{00}$.

Proof. Without loss of generality we may assume that $\text{supp}(f) \subseteq \sigma_n$. Let $v^* = \sum_{j \in A} \mathbf{e}_j^*$. Taking into account (3.9) and (3.2) we obtain

$$|\mathbf{v}_n^*(S_A(f))| = \frac{\Lambda_{|\sigma_n|}}{|\sigma_n|} |v^*(f)| \leq \frac{\Lambda_{|\sigma_n|}}{|\sigma_n|} \|v^*\|_{\mathbb{S}^*} \|f\|_{\mathbb{S}} \leq 2 \frac{\Lambda_{|\sigma_n|}}{|\sigma_n|} \frac{|A|}{\Lambda_{|A|}} \|f\|_{\mathbb{S}},$$

as desired. \square

For further reference let us write down the following easy result.

Lemma 3.12. *Let $\sigma = (\sigma_n)_{n=1}^{\infty}$ be an ordered partition of \mathbb{N} . Let Λ_n and \mathbf{v}_n^* for $n \in \mathbb{N}$ be as in (3.4) with $\mathbb{S} = \ell_1$. Then*

$$|\mathbf{v}_n^*(S_A(f))| \leq \frac{\Lambda_{|A|}}{\Lambda_{|\sigma_n|}} |\mathbf{v}_n^*(f)| + \sum_{j \in \sigma_n} |a_j - \text{Av}(f, \sigma_n)|$$

whenever $n \in \mathbb{N}$, $A \subseteq \sigma_n$, and $f = (a_j)_{j=1}^{\infty} \in c_{00}$.

Proof. We have $\Lambda_m = m$ for all m . Hence

$$\begin{aligned} |\mathbf{v}_n^*(S_A(f))| &= \left| \sum_{j \in A} a_j \right| \\ &\leq |A| |\text{Av}(f, \sigma_n)| + \left| \sum_{j \in A} a_j - \text{Av}(f, \sigma_n) \right| \\ &\leq |A| |\text{Av}(f, \sigma_n)| + \sum_{j \in \sigma_n} |a_j - \text{Av}(f, \sigma_n)| \\ &= \frac{\Lambda_{|A|}}{\Lambda_{|\sigma_n|}} |\mathbf{v}_n^*(f)| + \sum_{j \in \sigma_n} |a_j - \text{Av}(f, \sigma_n)|. \end{aligned}$$

□

Lemma 3.13. *Let $(\lambda_n)_{n=1}^\infty$ be a non-decreasing sequence of positive scalars such that $(\lambda_n/n)_{n=1}^\infty$ is non-increasing. Then*

$$\frac{k}{k+n}\lambda_n + \frac{n}{k+n}\lambda_k \leq 2\lambda_k$$

for all $k, n \in \mathbb{N}$.

Proof. In the case when $k \leq n$ we have

$$\frac{k}{k+n}\lambda_n + \frac{n}{k+n}\lambda_k \leq \frac{n}{k+n}\lambda_k + \frac{n}{k+n}\lambda_k = \frac{2n}{n+k}\lambda_k \leq 2\lambda_k,$$

and in the case $n \leq k$,

$$\frac{k}{k+n}\lambda_n + \frac{n}{k+n}\lambda_k \leq \frac{k}{k+n}\lambda_k + \frac{n}{k+n}\lambda_k = \lambda_k.$$

□

Lemma 3.14. *Let \mathbb{S} be a subsymmetric sequence space and $\sigma = (\sigma_n)_{n=1}^\infty$ be an ordered partition of \mathbb{N} . For all $A \subseteq \mathbb{N}$ and $f \in \mathbb{S}$ we have*

$$\|Q_\sigma(S_A(f))\|_{\mathbb{S}} \leq 5\|Q_\sigma(f)\|_{\mathbb{S}} + 2 \sum_{n=1}^\infty \frac{\Lambda_{|A_n|}}{\Lambda_{|\sigma_n|}} |\mathbf{v}_n^*(f)|,$$

where $A_n := A \cap \sigma_n$.

Proof. We write $\|\cdot\|$ rather than $\|\cdot\|_{\mathbb{S}}$ as there is no possible confusion in this proof. Put $B_n = \sigma_n \setminus A_n$. Note that for every $f \in Q_\sigma(\mathbb{S})$

$$\text{Av}(f, A_n)|A_n| + \text{Av}(f, B_n)|B_n| = 0,$$

and so $P_\sigma(S_A(f)) = (y_j)_{j=1}^\infty$, where

$$y_j = \begin{cases} \text{Av}(f, A_n) \frac{|A_n|}{|\sigma_n|} & \text{if } j \in A_n \\ -\text{Av}(f, B_n) \frac{|B_n|}{|\sigma_n|} & \text{if } j \in B_n. \end{cases}$$

Let P_τ be the averaging projection with respect to the partition $\tau = (A_n, B_n)_{n=1}^\infty$. From Theorem 3.1 we infer that $\|P_\tau\|_{\mathbb{S} \rightarrow \mathbb{S}} \leq 4$. Taking into account the lattice structure of \mathbb{S} we get

$$\|Q_\sigma(S_A(f))\| \leq \|P_\sigma(S_A(f))\| + \|S_A(f)\| \leq \|P_\tau(f)\| + \|f\| \leq 5\|f\|.$$

Assume that $f \in P_\sigma(\mathbb{S})$. Pick $(b_n)_{n=1}^\infty$ such that $\mathbf{e}_j^*(f) = b_n$ if $j \in \sigma_n$. Then $Q_\sigma(S_A(f)) = (c_j)_{j=1}^\infty$, where

$$c_j = \begin{cases} b_n \frac{|B_n|}{|\sigma_n|} & \text{if } j \in A_n \\ -b_n \frac{|A_n|}{|\sigma_n|} & \text{if } j \in B_n. \end{cases}$$

Therefore, using Lemma 3.13,

$$\begin{aligned} \|Q_\sigma(S_A(f))\| &\leq \sum_{n=1}^\infty |b_n| \left(\frac{|B_n|}{|\sigma_n|} \Lambda_{|A_n|} + \frac{|A_n|}{|\sigma_n|} \Lambda_{|B_n|} \right) \\ &\leq 2 \sum_{n=1}^\infty |b_n| \Lambda_{|A_n|} \\ &= 2 \sum_{n=1}^\infty \frac{\Lambda_{|A_n|}}{\Lambda_{|\sigma_n|}} |\mathbf{v}_n^*(f)|. \end{aligned}$$

We complete the proof by expressing any $f \in \mathbb{S}$ in the form $f_1 + f_2$ with $f_1 \in P_\sigma(\mathbb{S})$ and $f_2 \in Q_\sigma(\mathbb{S})$. \square

Lemma 3.15. *Let $(N_n)_{n=1}^\infty$ be a sequence of positive integers such that $\sum_{k=1}^n N_k \lesssim N_{n+1}$ for $n \in \mathbb{N}$, and $(\lambda_n)_{n=1}^\infty$ be a sequence of positive scalars having the LRP. Assume that $(\lambda_n)_{n=1}^\infty$ and $(n/\lambda_n)_{n=1}^\infty$ are non-decreasing. Then*

$$\sup_{r \in \mathbb{N}} \sum_{n=r}^\infty \frac{\lambda_{N_r}}{\lambda_{N_n}} < \infty.$$

Proof. Since $(\lambda_n)_{n=1}^\infty$ has the LRP, an appeal to [1, Theorem 2.12] yields $0 < \alpha < 1$ and $0 < C_1 < \infty$ such that

$$\frac{\lambda_n}{n^\alpha} \leq C_1 \frac{\lambda_m}{m^\alpha} \quad \text{for all } n \leq m.$$

Let $C_2 > 1$ be such that $M_r := \sum_{n=1}^r N_n \leq C_2 N_r$ for all $r \in \mathbb{N}$. Then, if $t = C_2/(1 + C_2)$ we have $M_{r+1} \leq t M_r$ for $r \in \mathbb{N}$. Hence

$$\begin{aligned} \sum_{n=r}^\infty \frac{\lambda_{N_r}}{\lambda_{N_n}} &\leq \sum_{j=n}^\infty \frac{M_n}{N_n} \frac{\lambda_{N_r}}{\lambda_{M_n}} \leq C_2 \sum_{n=r}^\infty \frac{\lambda_{M_r}}{\lambda_{M_n}} \leq C_1 C_2 \sum_{n=r}^\infty \left(\frac{M_r}{M_n} \right)^\alpha \\ &\leq C_1 C_2 \sum_{n=r}^\infty t^{\alpha(n-r)} = C_1 C_2 \frac{1}{1 - t^\alpha}, \end{aligned}$$

as desired. \square

Lemma 3.16. *Assume that all the hypotheses of Theorem 3.9 hold and that either $\mathbb{S} = \ell_1$ or $(\Lambda_n)_{n=1}^\infty$ has both the LRP and the URP. Then there exists a constant C_a such that*

$$\|S_A(f)\|_{\mathcal{B}, \mathbb{S}, \sigma} \leq C_a \|f\|_{\mathcal{B}, \mathbb{S}, \sigma} \quad \text{for all } f \in c_{00},$$

whenever A and r are such that $A \subset \cup_{n=r}^\infty \sigma_n$ and $|A| \leq M_r$.

Proof. Without loss of generality we suppose that $\mathcal{B} = (\mathbf{x}_n)_{n=1}^\infty$ is normalized and bimonotone. Put $A_n = A \cap \sigma_n$. There is $r \in \mathbb{N}$ such that $A_n = \emptyset$ for $1 \leq n \leq r-1$. Then,

$$\left\| \sum_{n=1}^\infty \mathbf{v}_n^*(S_A(f)) \mathbf{x}_n \right\|_{\mathbb{X}} \leq \sum_{n=1}^\infty |\mathbf{v}_n^*(S_A(f))| = \sum_{n=r}^\infty |\mathbf{v}_n^*(S_{A_n}(f))|.$$

In the case when $(\Lambda_n)_{n=1}^\infty$ has the URP, from inequality (3.9) and [1, Theorem 2.12] it follows that $(\Lambda_n^*)_{n=1}^\infty$ has the LRP. Using Lemma 3.15 and the fact that $(\Lambda_n^*)_{n=1}^\infty$ is doubling gives

$$C_1 := \sup_r \frac{\Lambda_{M_r}^*}{\Lambda_{|\sigma_r|}^*} < \infty \text{ and } C_2 = \sup_r \sum_{n=r}^\infty \frac{\Lambda_{|\sigma_r|}^*}{\Lambda_{|\sigma_n|}^*} < \infty.$$

Then, by Lemmas 3.3, 3.4, and 3.11,

$$\begin{aligned} \sum_{n=r}^\infty |\mathbf{v}_n^*(S_{A_n}(f))| &\leq 2 \sum_{n=r}^\infty \frac{\Lambda_{|A_n|}^*}{\Lambda_{|\sigma_n|}^*} \|S_{\sigma_n}(f)\|_{\mathbb{S}} \\ &\leq 2 \sum_{n=r}^\infty \frac{\Lambda_{M_r}^*}{\Lambda_{|\sigma_n|}^*} \|S_{\sigma_n}(f)\|_{\mathcal{B}, \mathbb{S}, \sigma} \\ &\leq 2C_1 \|f\|_{\mathcal{B}, \mathbb{S}, \sigma} \sum_{n=r}^\infty \frac{\Lambda_{|\sigma_r|}^*}{\Lambda_{|\sigma_n|}^*} \\ &\leq 2C_1 C_2 \|f\|_{\mathcal{B}, \mathbb{S}, \sigma}. \end{aligned}$$

In the case when $\mathbb{S} = \ell_1$, Lemma 3.12 gives

$$\begin{aligned} \sum_{n=r}^\infty |\mathbf{v}_n^*(S_{A_n}(f))| &\leq \sum_{n=r}^\infty \frac{\Lambda_{|A_n|}}{\Lambda_{|\sigma_n|}} |\mathbf{v}_n^*(f)| + \sum_{n=r}^\infty \sum_{j \in \sigma_n} |a_j - \text{Av}(f, \sigma_n)| \\ &\leq \|Q_\sigma(f)\|_1 + \sum_{n=r}^\infty \frac{\Lambda_{|A_n|}}{\Lambda_{|\sigma_n|}} |\mathbf{v}_n^*(f)| \end{aligned}$$

The sequence $(\Lambda_n)_{n=1}^\infty$ is also doubling, and our hypothesis always gives that it has the LRP. Hence, appealing again to Lemma 3.15,

$$C_3 := \sup_r \frac{\Lambda_{M_r}}{\Lambda_{|\sigma_r|}} < \infty \text{ and } C_4 = \sup_r \sum_{n=r}^\infty \frac{\Lambda_{|\sigma_r|}}{\Lambda_{|\sigma_n|}} < \infty.$$

Consequently,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\Lambda_{|A_n|}}{\Lambda_{|\sigma_n|}} |\mathbf{v}_n^*(f)| &\leq \sum_{n=r}^{\infty} \frac{\Lambda_{M_r}}{\Lambda_{|\sigma_n|}} |\mathbf{v}_n^*(f)| \\ &\leq C_3 \sum_{n=r}^{\infty} \frac{\Lambda_{|\sigma_r|}}{\Lambda_{|\sigma_n|}} |\mathbf{v}_n^*(f)| \\ &\leq C_3 C_4 \left\| \sum_{n=1}^{\infty} \mathbf{v}_n^*(f) \mathbf{x}_n \right\|_{\mathbb{X}}. \end{aligned}$$

Using Lemma 3.14 and combining we get the desired result with $C_a = 2C_1C_2 + \max\{5, 2C_3C_4\}$ in the case when $(\Lambda_n)_{n=1}^{\infty}$ has both the LRP and the URP, and $C_a = \max\{6, 3C_1C_2\}$ when $\mathbb{S} = \ell_1$. \square

Theorem 3.17 (cf. [16, Theorem 7.1]). *Assume that all the hypotheses of Theorem 3.9 hold and that either $\mathbb{S} = \ell_1$ or $(\Lambda_n)_{n=1}^{\infty}$ has both the LRP and the URP. Then the unit-vector system is an almost greedy basis for $\mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma]$ with fundamental function equivalent to $(\Lambda_n)_{n=1}^{\infty}$.*

Proof. Without loss of generality we assume that \mathcal{B} is bi-monotone. Notice that, put in other words, Corollary 3.10 says that for all $f \in c_{00}$,

$$\frac{1}{C_1} \|f\|_{d_1^{\infty}(\mathbf{w})} \leq \|f\|_{\mathcal{B}, \mathbb{S}, \sigma} \leq C_2 \|f\|_{d_1(\mathbf{w})},$$

for some positive constants C_1, C_2 . Hence it suffices to prove that \mathcal{E} is a quasi-greedy basis for $\mathbb{Y} := \mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma]$.

Let C_a be as in Lemma 3.16. By Theorem 3.6 (a),

$$C_b := \sup_{m \in \mathbb{N}} \|\text{Id}_{\mathbb{F}^{\mathbb{N}}} - S_m\|_{\mathbb{Y} \rightarrow \mathbb{Y}} < \infty.$$

Let $f = (a_j)_{j=1}^{\infty} \in c_{00}$ and let $F \subseteq \mathbb{N}$ be a non-empty set such that $|a_j| \leq |a_k|$ whenever $k \in F$ and $j \in \mathbb{N} \setminus F$. Set $m = |F|$ and pick $r \in \mathbb{N}$ such that $m \in \sigma_r$. Let $A = [1, m] \setminus F$ and $B = F \cap [m+1, \infty)$. We have

$$F \cup A = \{1, \dots, m\} \cup B, \quad F \cap A = \{1, \dots, m\} \cap B = \emptyset.$$

Therefore $s := |A| = |B| \leq m \leq M_r$, $B \subseteq \cup_{n=r}^{\infty} \sigma_n$ and

$$S_F(f) = S_m(f) - S_A(f) + S_B(f).$$

We deduce that $\|S_B(f)\|_{\mathcal{B}, \mathbb{S}, \sigma} \leq C_a \|f\|_{\mathcal{B}, \mathbb{S}, \sigma}$ and that, if $(a_n^*)_{n=1}^{\infty}$ is the non-increasing rearrangement of f , $|a_j| \leq a_s^*$ for all $j \in A$. Then,

$$\|S_A(f)\|_{\mathcal{B}, \mathbb{S}, \sigma} \leq C_2 \max_{j \in A} |a_j| \Lambda_s \leq a_s^* \Lambda_r \leq C_1 C_2 \|f\|_{\mathcal{B}, \mathbb{S}, \sigma}.$$

Combining we get

$$\begin{aligned} \|f - S_F(f)\|_{\mathcal{B}, \mathbb{S}, \sigma} &\leq \|f - S_k(f)\|_{\mathcal{B}, \mathbb{S}, \sigma} + \|S_A(f)\|_{\mathcal{B}, \mathbb{S}, \sigma} + \|S_B(f)\|_{\mathcal{B}, \mathbb{S}, \sigma} \\ &\leq (C_b + C_a + C_1 C_2) \|f\|_{\mathcal{B}, \mathbb{S}, \sigma}. \end{aligned}$$

That is, the unit vector system is $(C_b + C_a + C_1 C_2)$ -quasi-greedy. \square

Remark 3.18. Note that from (3.11) it follows that

$$\log \left(\sum_{n=1}^r |\sigma_n| \right) \gtrsim r, \quad r \in \mathbb{N}. \quad (3.12)$$

This exponential growth is essentially optimal. Indeed, Theorem 1.1(a) implies that when the DKK-method is applied to a basis \mathcal{B} for a Banach space \mathbb{X} with $L_m[\mathcal{B}, \mathbb{X}] \approx m$ (such as the summing basis of c_0) it can only produce a Banach space $\mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma]$ for which the unit-vector basis is quasi-greedy when (3.12) holds.

4. BANACH SPACES HAVING QUASI-GREEDY BASES WITH LARGE CONDITIONALITY CONSTANTS

The conductive thread of this section is the search for results that will allow us to include the spaces $Z_{p,q}$, $B_{p,q}$, and $D_{p,q}$ (see Section 1 for the definitions) in the list of Banach spaces possessing highly conditional quasi-greedy bases. We recall that the matrix spaces $Z_{p,q}$ are isomorphic to Besov spaces over Euclidean spaces (see, e.g., [2]) and that the mixed-norm spaces $B_{p,q}$ are isomorphic to Besov spaces over the unit interval (see, e.g., [3, Appendix 4.2]).

Apart from the trivial cases, namely

$$D_{q,q} \approx Z_{q,q} \approx B_{q,q} \approx \ell_q, \quad 1 \leq q < \infty,$$

and the case

$$\ell_q \approx B_{2,q}, \quad 1 < q < \infty, \quad (4.1)$$

all the above-mentioned spaces are mutually non-isomorphic (see [3]). The isomorphism in (4.1) was obtained by Pełczyński in [35] by combining the uniform complemented embeddings

$$\ell_2^n \lesssim_c \ell_p^{2^n} \text{ for } n \in \mathbb{N}, \text{ if } 1 < p < \infty, \quad (4.2)$$

(which can be obtained as a consequence of the boundedness of the Rademacher projections in L_p) with the Pełczyński decomposition technique (see, e.g., [8, Theorem 2.2.3]). Another well-known consequence of Pełczyński's decomposition technique is that for any unbounded sequence of integers $(d_n)_{n=1}^\infty$ we have

$$B_{p,q} \approx (\oplus_{n=1}^\infty \ell_p^{d_n})_q, \quad p \in [1, \infty], \quad q \in \{0\} \cup [1, \infty), \quad (4.3)$$

(see, e.g., [3, Appendix 4.1].)

Theorem 4.1. *Let \mathbb{X} be a Banach space with a basis \mathcal{B} and suppose that either $\mathbb{S} = \ell_1$ or \mathbb{S} is a subsymmetric sequence space with nontrivial type. Assume that $\mathbb{S} \lesssim_c \mathbb{X}$ and that $L_m[\mathcal{B}] \gtrsim \delta(m)$ for $m \in \mathbb{N}$, for some doubling non-decreasing function $\delta: [0, \infty) \rightarrow [0, \infty)$. Then there is an almost greedy basis \mathcal{B}_κ for \mathbb{X} with fundamental function equivalent to $(\|\sum_{j=1}^n \mathbf{e}_j\|_{\mathbb{S}})_{n=1}^\infty$, and with $L_m[\mathcal{B}_\kappa] \gtrsim \delta(\log m)$ for $m \in \mathbb{N}$.*

Proof. For each $n \in \mathbb{N}$ put $\sigma_n = [2^{n-1}, 2^n - 1]$, and let \mathbf{v}_n be defined as in (3.4). Notice that $\sigma = (\sigma_n)_{n=1}^\infty$ verifies both (3.11) and (3.7) and that $\mathcal{V} = (\mathbf{v}_n)_{n=1}^\infty$ is an unconditional basis for $P_\sigma[\mathbb{S}]$. Then, by [17, Proposition 4.1], Lemma 2.2, Theorem 3.17 and Proposition 3.8, the unit-vector system is an almost greedy basis as desired for $\mathbb{Y} := \mathbb{Y}[\mathcal{V} \oplus \mathcal{B}, \mathbb{S}, \sigma]$. Let \mathbb{Z} such that $\mathbb{X} \approx \mathbb{S} \oplus \mathbb{Z}$. The chain of isomorphisms

$$\mathbb{Y} \approx Q_\sigma(\mathbb{S}) \oplus P_\sigma(\mathbb{S}) \oplus \mathbb{X} \approx \mathbb{S} \oplus \mathbb{S} \oplus \mathbb{Z} \approx \mathbb{S} \oplus \mathbb{Z} \approx \mathbb{X}$$

(see Remark 3.2) completes the proof. \square

Remark 4.2. As the attentive reader may have noticed, the only information we use about the subsymmetric space \mathbb{S} in the proof above in the case when $\mathbb{S} \neq \ell_1$ is that the fundamental function of its canonical basis has both the LRP and the URP. Thus, Theorem 4.1 holds replacing the hypothesis “ \mathbb{S} has nontrivial type” with the hypothesis “the fundamental function of the subsymmetric basis of \mathbb{S} has both the LRP and the URP.”

In order to effectively use Theorem 4.1 we need to ensure the existence of bases \mathcal{B} with large conditionality constants $(L_m[\mathcal{B}])_{m=1}^\infty$ in subsymmetric sequence spaces. We start by recalling the following result by Garrigós and Wojtaszczyk, which, in our language, reads as follows.

Theorem 4.3 (cf. [24, Proposition 3.10]). *For each $0 < a < 1$ there is a basis \mathcal{B} in ℓ_2 with $L_m[\mathcal{B}] \gtrsim m^a$ for $m \in \mathbb{N}$.*

Proposition 4.4. *For each $0 < a < 1$ and each $1 < q < \infty$ there is a basis \mathcal{B} in ℓ_q with $L_m[\mathcal{B}] \gtrsim m^a$ for $m \in \mathbb{N}$.*

Proof. Apply Theorem 4.3 for picking a basis $\mathcal{B}_0 = (\mathbf{x}_j)_{j=1}^\infty$ for ℓ_2 with $L_m[\mathcal{B}_0] \gtrsim m^a$ for $m \in \mathbb{N}$. By Lemma 2.3, $\mathcal{B} = \bigoplus_{n=1}^\infty (\mathbf{x}_j)_{j=1}^{2^n}$ is a basis for $\mathbb{X} = (\bigoplus_{n=1}^\infty \ell_2^{(2^n)}[\mathcal{B}_0])_p$ with $L_m[\mathcal{B}] \gtrsim m^a$. Since any N -dimensional Hilbert space is isometric to ℓ_2^N , the isomorphisms (4.1) and (4.3) yield $\mathbb{X} \approx (\bigoplus_{n=1}^\infty \ell_2^{(2^n)})_q \approx B_{2,q} \approx \ell_q$. \square

Our next result improves Corollary 3.13 from [24], where it is shown the existence of quasi-greedy bases as conditional as possible in ℓ_q , $1 < q < \infty$. The main improvement consists of building, for $q \neq 2$, almost greedy bases instead of quasi-greedy ones.

Theorem 4.5 (cf. [24, Theorem 1.2, Corollary 3.12 and Corollary 3.13]). *Let \mathbb{X} be a Banach space with a basis and $1 < q < \infty$. If $\ell_q \lesssim_c \mathbb{X}$ then for any $0 < a < 1$ the space \mathbb{X} has an almost greedy basis \mathcal{B}_κ with fundamental function equivalent to $(m^{1/q})_{m=1}^\infty$ and with $L_m[\mathcal{B}_\kappa] \gtrsim (\log m)^a$ for $m \in \mathbb{N}$.*

Proof. By Lemma 2.2 and Proposition 4.4, $\ell_p \oplus \mathbb{X}$ has a basis \mathcal{B} with $L_m[\mathcal{B}] \gtrsim m^a$ for $m \in \mathbb{N}$. Then, Theorem 4.1 gives a basis as desired for $\ell_q \oplus \mathbb{X}$. Finally, since $\ell_q \oplus \ell_q \approx \ell_q$, we infer that $\ell_q \oplus \mathbb{X} \approx \mathbb{X}$. \square

Example 4.6.

- (i) Theorem 4.5 applies to L_q and to ℓ_q for $1 < q < \infty$. More generally, Theorem 4.5 yields that if $q \in (1, \infty)$, any separable \mathcal{L}_q -space \mathbb{X} has almost greedy bases as conditional as possible whose fundamental function is equivalent to $(m^{1/q})_{m=1}^\infty$ (see [31, Proposition 7.3] and [27, Theorem 5.1]). Moreover, if \mathbb{X} is not isomorphic to ℓ_q it also has almost greedy bases as conditional as possible whose fundamental function is equivalent to $(m^{1/2})_{m=1}^\infty$ (see [26, Corollary 1] and [29, Corollary 1]).
- (ii) Let $1 < p, q < \infty$. Theorem 4.5 applies to the superreflexive spaces $Z_{p,q}$, $B_{p,q}$ and $D_{p,q}$.
- (iii) Theorem 4.5 also applies to Lorentz sequence spaces. In fact, appealing also to Theorem 4.1 we claim that given $\mathbf{w} = (w_n)_{n=1}^\infty$ non-increasing, $1 < q < \infty$, and $\epsilon > 0$, the space $d_q(\mathbf{w})$ has almost greedy bases \mathcal{B}_κ with $L_m[\mathcal{B}_\kappa] \gtrsim (\log m)^{1-\epsilon}$ and with fundamental function equivalent either to $(m^{1/q})_{m=1}^\infty$ or to $(W_m^{1/q})_{m=1}^\infty$, where $W_m = \sum_{n=1}^m w_n$. In the case when $(W_m)_{m=1}^\infty$ has the LRP, the space $d_q(\mathbf{w})$ is superreflexive (see [9, Theorem 1]) and, then, the estimate of $L_m[\mathcal{B}_\kappa]$ is sharp. In particular, the classical Lorentz sequence spaces $\ell_{p,q}$ for $1 < p, q < \infty$ have almost greedy bases as conditional as possible with fundamental function equivalent either to $(m^{1/q})_{m=1}^\infty$ or to $(m^{1/p})_{m=1}^\infty$.

Next we make an intermission within our main subject to present a more abstract application of Theorem 4.1.

Theorem 4.7. *Let \mathbb{S} be a subsymmetric sequence space with nontrivial type. Then, for every $0 < a < 1$, \mathbb{S} contains an almost greedy basic sequence $\mathcal{B}_\kappa = (\mathbf{x}_n)_{n=1}^\infty$ such that $L_m[\mathcal{B}_\kappa] \gtrsim (\log m)^a$ for $m \in \mathbb{N}$.*

Proof. Combining Mazur's construction of basic sequences (see e.g., [7, Proposition 3.1]) with Dvoretzky's theorem on the finite representability of ℓ_2 in all infinite-dimensional Banach spaces [22] it follows that every infinite-dimensional Banach space contains a basic sequence $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$ such that $[\mathbf{x}_j]_{j=2^n}^{j=2^{n+1}-1}$ is 2-isomorphic to $\ell_2^{2^n}$ for all $n \geq 1$. Moreover, Theorem 4.3 allows us to select the basis vectors $[\mathbf{x}_j]_{j=2^n}^{j=2^{n+1}-1}$ in such a way that we may ensure that $L_m[\mathcal{B}] \gtrsim m^a$ for $m \in \mathbb{N}$. Choose such a basic sequence \mathcal{B} inside \mathbb{S} and let \mathbb{X} be its closed linear span. Setting $\sigma = ([2^{n-1}, 2^n - 1])_{n=1}^\infty$, by Theorem 4.1 the unit-vector basis \mathcal{E} is an almost greedy basis for $\mathbb{Y} = \mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma]$ satisfying $L_m[\mathcal{E}, \mathbb{Y}] \gtrsim (\log m)^a$ for $m \in \mathbb{N}$. Finally, by Theorem 3.6 (c) and Remark 3.2,

$$\mathbb{Y}[\mathcal{B}, \mathbb{S}, \sigma] \approx Q_\sigma(\mathbb{S}) \oplus \mathbb{X} \subseteq \mathbb{S} \oplus \mathbb{S} \approx \mathbb{S}.$$

□

Remark 4.8. It follows from Theorem 4.7 that if \mathbb{X} has nontrivial type then every spreading model of \mathbb{X} generated by a weakly null sequence contains a basic sequence satisfying the conclusion of Theorem 4.7. This is so because spreading models of \mathbb{X} are finitely represented in \mathbb{X} (and hence have nontrivial type) and those generated by weakly null sequences are subsymmetric.

Example 4.6 shows that Theorem 4.1 is strong enough for a wide class of superreflexive Banach spaces. When dealing with non-superreflexive Banach spaces we need to combine the DKK-method with other techniques. To that end, we recover some ideas from [7]. Recall that a basis $(\mathbf{x}_j)_{j=1}^\infty$ is said to be of *type P* if

$$\sup_k \left\| \sum_{j=1}^k \mathbf{x}_j \right\| < \infty \quad \text{and} \quad \inf_j \|\mathbf{x}_j\| > 0.$$

Notice that both the unit-vector system in c_0 and the *difference system* $\mathcal{D} = (\mathbf{d}_j)_{j=1}^\infty$ in ℓ_1 , defined by

$$\mathbf{d}_j = \mathbf{e}_j - \mathbf{e}_{j-1} \quad (\text{with the convention } \mathbf{e}_0 = 0),$$

are bases of type P. It is known that the *summing system* $\mathcal{S} = (\mathbf{s}_j)_{j=1}^\infty$, given by

$$\mathbf{s}_j = \sum_{k=1}^j \mathbf{e}_k.$$

is a conditional basis for c_0 . Most proofs of this fact (see, e.g., [8, Example 3.1.2]) give the following.

Lemma 4.9. *The summing system \mathcal{S} is a basis for c_0 with $L_m[\mathcal{S}, c_0] \approx m$ for $m \in \mathbb{N}$, and $c_0^{(N)}[\mathcal{S}] = \ell_\infty^N$ for all $N \in \mathbb{N}$.*

By duality, the difference system is a conditional basis for ℓ_1 . Indeed, we have the following.

Lemma 4.10. *The difference system \mathcal{D} is a basis of ℓ_1 with $L_m[\mathcal{D}, \ell_1] \approx m$ for $m \in \mathbb{N}$, and $\ell_1^{(N)}[\mathcal{D}] = \ell_1^N$ for all $N \in \mathbb{N}$.*

Lemmas 4.9 and 4.10 exhibit the fact that c_0 and ℓ_1 have bases as conditional as possible. The following lemma shows that Banach spaces with a basis of type P follow the pattern of c_0 and ℓ_1 .

Lemma 4.11 (see [7]). *Let \mathcal{B}_0 be a basis of type P of a Banach space \mathbb{X} . Then there is a basis \mathcal{B} for \mathbb{X} such that $L_m[\mathcal{B}] \approx m$ for $m \in \mathbb{N}$ and $\mathbb{X}^{(2^n-2)}[\mathcal{B}] = \mathbb{X}^{(2^n-2)}[\mathcal{B}_0]$ for all $n \geq 2$.*

Proof. Although it is not explicitly stated, the proof of Theorem 3.3 from [7] gives the result. \square

Theorem 4.12. *Suppose \mathbb{X} is a Banach space with a basis of type P, and let $(\mathbb{S}, \|\cdot\|_{\mathbb{S}})$ be a subsymmetric Banach space. Assume that $\mathbb{S} \lesssim_c \mathbb{X}$ and that \mathbb{S} has nontrivial type. Then \mathbb{X} has an almost greedy basis \mathcal{B}_κ whose fundamental function is equivalent to $(\|\sum_{j=1}^m \mathbf{e}_j\|_{\mathbb{S}})_{m=1}^\infty$ and such that $L_m[\mathcal{B}_\kappa] \approx \log m$ for $m \geq 2$.*

Proof. Just combine Lemma 4.11 with Theorem 4.1. \square

Example 4.13.

- (i) By Proposition 2.4 and [7, Proposition 2.10] (which states that the unit-vector system is a basis of type P for Pisier-Xu spaces), Theorem 4.12 applies to $\mathcal{W}_{p,q}^0$ for $1 < p, q < \infty$.
- (ii) The unit-vector system is a shrinking basis of type P for the James space $\mathcal{J}^{(p)}$. It is also known that $\ell_p \lesssim_c \mathcal{J}^{(p)}$. Actually, the linear operator L defined in (2.1) is bounded from ℓ_p into $\mathcal{J}^{(p)}$ and the linear operator T defined in (2.2) is bounded from $\mathcal{J}^{(p)}$ into ℓ_p (see also [14, Lemma 3.2]). So, Theorem 4.12 applies both to $\mathcal{J}^{(p)}$ and $(\mathcal{J}^{(p)})^*$.

Theorem 4.14. *Suppose \mathbb{X} is a Banach space with a basis and that $(\mathbb{S}, \|\cdot\|_{\mathbb{S}})$ is a subsymmetric Banach space. Assume that $\mathbb{S} \lesssim_c \mathbb{X}$, that \mathbb{S} has nontrivial type, and that either $\ell_1 \lesssim_c \mathbb{X}$ or $c_0 \lesssim_c \mathbb{X}$. Then \mathbb{X} has an almost greedy basis \mathcal{B}_κ whose fundamental function is equivalent to $(\|\sum_{j=1}^m \mathbf{e}_j\|_{\mathbb{S}})_{m=1}^\infty$ and such that $L_m[\mathcal{B}_\kappa] \approx \log m$ for $m \geq 2$.*

Theorem 4.15. *Let \mathbb{X} be a Banach space with a basis. If $\ell_1 \lesssim_c \mathbb{X}$ then \mathbb{X} has an almost greedy basis \mathcal{B}_κ with fundamental function equivalent to $(m)_{m=1}^\infty$ and $L_m[\mathcal{B}_\kappa] \approx \log m$ for $m \geq 2$.*

Proof of Theorems 4.14 and 4.15. Since $\ell_1 \oplus \ell_1 \approx \ell_1$ and $c_0 \oplus c_0 \approx c_0$, we infer from Lemma 4.10 (or Lemma 4.9) and Lemma 2.2 that \mathbb{X} has a basis \mathcal{B} with $L_m[\mathcal{B}] \approx m$ for $m \in \mathbb{N}$. Appealing to Theorem 4.1 (we put $\mathbb{S} = \ell_1$ when proving Theorem 4.15) concludes the proof. \square

Example 4.16.

- (i) Given $1 < p < \infty$, Theorem 4.14 applies to the spaces $D_{p,0}$, $D_{p,1}$, $Z_{1,p}$, $Z_{p,1}$, $Z_{p,0}$, $Z_{0,p}$, and the fundamental function of the bases we obtain is $(m^{1/p})_{m=1}^\infty$.
- (ii) Theorem 4.14 also applies to the Hardy space H_1 and its predual VMO. Indeed, these spaces have complemented Hilbertian subspaces (see, e.g., [34]) and so the fundamental functions of the almost greedy bases we obtain are equivalent to $(m^{1/2})_{m=1}^\infty$.

Example 4.17.

- (i) The list of Banach spaces for which Theorem 4.15 applies includes $D_{1,p}$, $Z_{p,1}$ and $Z_{1,p}$ for $p \in \{0\} \cup (1, \infty)$, $B_{p,1}$ for $p \in (1, \infty]$, ℓ_1 , L_1 , the Hardy space H_1 , and the Lorentz sequence spaces $d_1(\mathbf{w})$ for \mathbf{w} decreasing.
- (ii) By invoking [31, Proposition 7.3] and [27, Theorem 5.1], Theorem 4.15 applies to any separable \mathcal{L}_1 -space. In this case, since \mathcal{L}_1 -spaces are GT-spaces (see [31, Theorem 4.1]), the democracy of the bases we obtain is satisfied automatically due to [20, Theorem 4.2].

At this point, the only reflexive spaces in the families $D_{p,q}$, $Z_{p,q}$ or $B_{p,q}$ that are not yet in our list of Banach spaces possessing almost greedy bases as conditional as possible are the Besov spaces $B_{1,q}$ and $B_{\infty,q}$ for $1 < q < \infty$. The following theorem takes care of them.

Theorem 4.18. *Suppose \mathbb{X} is a Banach space with a basis such that either $B_{\infty,q} \lesssim_c \mathbb{X}$ or $B_{1,q} \lesssim_c \mathbb{X}$ for some $1 < q < \infty$. Then \mathbb{X} has an almost greedy basis \mathcal{B}_κ with fundamental function equivalent to $(m^{1/q})_{m=1}^\infty$ and with $L_m[\mathcal{B}_\kappa] \approx \log m$ for $m \geq 2$.*

Proof. Let $p \in \{1, \infty\}$. Since $\ell_q \lesssim_c \mathbb{X}$ and $\ell_q \oplus B_{p,q} \approx B_{p,q}$, \mathbb{X} has an almost greedy basis whose fundamental function is equivalent to $(m^{1/q})_{m=1}^\infty$, and we have $\mathbb{X} \approx B_{p,q} \oplus \mathbb{X}$. Hence, taking into account Lemma 2.2, it suffices to prove the result for $\mathbb{X} = B_{p,q}$. Let \mathcal{B} be the difference basis in ℓ_1 for $p = 1$ and the summing basis of c_0 for

$p = \infty$. Set $\sigma = ([2^{n-1}, 2^n - 1])_{n=1}^\infty$. By Lemma 2.3 and Theorem 4.1, the sequence $\bigoplus_{n=1}^\infty (\mathbf{e}_j)_{j=1}^{2^n-1}$ is an almost greedy basis as desired for $\mathbb{Z} = (\bigoplus_{n=1}^\infty \mathbb{Y}^{(2^n-1)}[\mathcal{B}, \ell_q, \sigma])_q$. By Theorem 3.6 (c),

$$\mathbb{Z} \approx \left(\bigoplus_{n=1}^\infty Q_\sigma(\ell_q^{2^n-1}) \oplus \ell_p^n \right)_q \approx \mathbb{V} \oplus B_{p,q},$$

where

$$\mathbb{V} = \left(\bigoplus_{n=1}^\infty Q_\sigma(\ell_q^{2^n-1}) \right)_q.$$

Notice that $\mathbb{V} \lesssim_c (\bigoplus_{n=1}^\infty \ell_q^{2^n-1})_q \approx \ell_q$. Moreover, appealing to (4.3) gives $\ell_q(B_{p,q}) \approx B_{p,q}$. Hence, applying Pełczyński's decomposition technique yields $\mathbb{Z} \approx B_{p,q}$. \square

Remark 4.19. By Propositions 4.1 and 4.4 from [17], the bases obtained in Theorems 4.5, 4.12, 4.14 and 4.18 are bi-democratic. Then (see [17, Theorem 5.4]) their dual bases also are almost greedy.

Now, in order to complete our study, it remains to deal with non-reflexive Besov spaces $B_{p,0}$. First, we realize that in this case it is hopeless to try to obtain almost greedy bases.

Theorem 4.20. *Let $1 \leq p < \infty$. Then $B_{p,0}$ has no superdemocratic basis. In particular, $B_{p,0}$ has no almost greedy basis.*

This will follow immediately from our next Proposition, taking into account that $B_{p,0}$ and c_0 are not isomorphic.

Proposition 4.21. *Let $(\mathbb{X}_n)_{n=1}^\infty$ be a sequence of finite-dimensional Banach spaces, and let \mathcal{B} be a superdemocratic basic sequence in $\mathbb{X} = (\bigoplus_{n=1}^\infty \mathbb{X}_n)_q$ for some $q \in \{0\} \cup [1, \infty)$. Then*

- (a) *If $q \neq 0$ the fundamental function of \mathcal{B} is equivalent to $(n^{1/q})_{n=1}^\infty$.*
- (b) *If $q = 0$ then \mathcal{B} is equivalent to the unit vector basis of c_0 .*

Proof. Given $N \in \mathbb{N}$, let $P_N: \mathbb{X} \rightarrow \mathbb{X}$ be the canonical projection onto the first N coordinates. With the convention $P_0 = 0$, put also $P_{M,N} = P_N - P_M$ and $P_{M,N}^c = \text{Id}_{\mathbb{X}} - P_{M,N}$ for $0 \leq M \leq N$. Let $\mathcal{B} = (\mathbf{x}_j)_{j=1}^\infty$ be a superdemocratic basic sequence in \mathbb{X} . Since the set $\{P_N(\mathbf{x}_j): j \in \mathbb{N}\}$ is compact, we have that for every $N \in \mathbb{N}$ every subsequence of \mathcal{B} possesses a further subsequence, say $(\mathbf{z}_j)_{j=1}^\infty$, such that $(P_N(\mathbf{z}_j))_{j=1}^\infty$ converges, and so $\lim_j P_N(\mathbf{z}_{2j}) - P_N(\mathbf{z}_{2j-1}) = 0$. Fix a sequence $(\varepsilon_k)_{k=1}^\infty$ of positive scalars. Applying the gliding-hump technique, we infer that there are increasing sequences $(j_k)_{k=1}^\infty$ and $(N_k)_{k=1}^\infty$ of positive integers such that

$$\left\| P_{N_{k-1}, N_k}^c(\mathbf{x}_{j_{2k}} - \mathbf{x}_{j_{2k-1}}) \right\| \leq \varepsilon_k,$$

Then, by the principle of small perturbations (see [8, Theorem 1.3.9]), if we choose $(\varepsilon_k)_{k=1}^\infty$ conveniently and put

$$\mathbf{y}_k = P_{N_{k-1}, N_k}(\mathbf{x}_{j_{2k}} - \mathbf{x}_{j_{2k-1}}), \quad k \in \mathbb{N},$$

the basic sequences $(\mathbf{y}_k)_{k=1}^\infty$ and $(\mathbf{x}_{j_{2k}} - \mathbf{x}_{j_{2k-1}})_{k=1}^\infty$ are equivalent. In particular, $(\mathbf{y}_k)_{k=1}^\infty$ is semi-normalized and, since it is disjointly supported, it is equivalent to the canonical basis of ℓ_q (c_0 in the case when $q = 0$). Therefore

$$\left\| \sum_{k=1}^m \mathbf{x}_{j_{2k}} - \sum_{k=1}^m \mathbf{x}_{j_{2k-1}} \right\| \approx \left\| \sum_{k=1}^m \mathbf{y}_k \right\| \approx m^{1/q}, \quad m \in \mathbb{N}$$

(with the usual modification when $q = 0$). Since $\phi_m[\mathcal{B}] \leq \phi_{2m}[\mathcal{B}] \leq 2\phi_m[\mathcal{B}]$, the superdemocracy of \mathcal{B} yields $\phi_m[\mathcal{B}] \approx m^{1/q}$ for $m \in \mathbb{N}$ in the case when $q \geq 1$ and $\phi_m[\mathcal{B}] \approx 1$ for $m \in \mathbb{N}$ in the case when $q = 0$. The proof of (a) is over, and from here (b) is straightforward. \square

Theorem 4.22. *Let \mathbb{X} be a Banach space with a quasi-greedy basis. Assume that $B_{p,0} \lesssim_c \mathbb{X}$ for some $1 \leq p < \infty$. Then \mathbb{X} has a quasi-greedy basis \mathcal{B}_κ with $L_m[\mathcal{B}_\kappa] \approx \log m$ for $m \geq 2$.*

Proof. The relations (4.2) and (4.3) give $B_{2,0} \lesssim_c B_{p,0}$ for $1 < p < \infty$. Moreover, since $B_{p,0} \approx B_{p,0} \oplus B_{p,0}$ we have $B_{p,0} \oplus \mathbb{X} \approx \mathbb{X}$. By Lemma 2.2 it suffices show the existence of a quasi-greedy basis as conditional as possible in the space $B_{p,0}$ in the case when $p \in \{1, 2\}$. Set $\sigma = ([2^{n-1}, 2^n - 1])_{n=1}^\infty$ and let \mathcal{S} be the summing basis of c_0 . Combining Lemma 4.9, Lemma 2.3 and Theorem 4.1 we obtain that $\mathcal{B}_\kappa = (\bigoplus_{n=1}^\infty (\mathbf{e}_j)_{j=1}^{2^n-1})_0$ is a quasi-greedy basis for the space

$$\mathbb{Z} := \left(\bigoplus_{n=1}^\infty \mathbb{Y}^{(2^n-1)}[\mathcal{S}, \ell_p, \sigma] \right)_0$$

with $L_m[\mathcal{B}_\kappa] \approx \log m$ for $m \geq 2$. By Theorem 3.6 (c), if we put

$$\mathbb{V} = \left(\bigoplus_{n=1}^\infty Q_\sigma(\ell_p^{2^n-1}) \right)_0,$$

we have

$$\mathbb{Z} \approx \mathbb{V} \oplus (\bigoplus_{n=1}^\infty \ell_\infty^n)_0 \approx \mathbb{V} \oplus c_0 \approx \mathbb{V}.$$

In the case when $p = 2$ we have $Q_\sigma(\ell_q^{2^n-1}) \approx \ell_2^{2^n-n-1}$ for $n \in \mathbb{N}$. Assume that $p = 1$ and let $B = \mathbb{N} \setminus \{2^n - 1 : n \in \mathbb{N}\}$. The coordinate projection S_B restricts to an isomorphism from $Q_\sigma(\ell_1)$ onto

$$\mathbb{W} = \{(a_n)_{n=1}^\infty \in \ell_1 : a_{2^n-1} = 0 \text{ for all } n \in \mathbb{N}\}.$$

Hence, $Q_\sigma(\ell_1^{2^n-1}) \approx \ell_1^{2^n-n-1}$ for $n \in \mathbb{N}$. In both cases, taking into account (4.3), we have $\mathbb{V} \approx (\bigoplus_{n=1}^\infty \ell_p^{2^n-n-1})_0 \approx B_{p,0}$. \square

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